

# SELF-CONSISTENT SEPARABLE RPA APPROACH FOR SKYRME FORCES: AXIAL NUCLEI

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## Abstract

The self-consistent separable RPA (random phase approximation) method is formulated for Skyrme forces with pairing. The method is based on a general self-consistent procedure for factorization of the two-body interaction. It is relevant for various density- and current-dependent functionals. The contributions of the time-even and time-odd Skyrme terms as well as of the Coulomb and pairing terms to the residual interaction are taken self-consistently into account. Most of the expression have a transparent analytical form, which makes the method convenient for the treatment and analysis. The separable character of the residual interaction allows to avoid diagonalization of high-rank RPA matrices and thus to minimize the calculation effort. The previous studies have demonstrated high numerical accuracy and efficiency of the method for spherical nuclei. In this contribution, the method is specified for axial nuclei. We provide systematic and detailed presentation of formalism and discuss different aspects of the model.

## I. INTRODUCTION

Effective nuclear forces (Skyrme, Gogny, ...) are widely used for description of diverse properties of atomic nuclei (see, for review [1]). However, their application to nuclear dynamics is still limited even in the linear regime which is usually treated within the random-phase-approximation (RPA). The theory is plagued by dealing with high-rank matrices which make the computations quite expensive. This is especially the case for non-spherical systems with their demanding configuration space. The rank of the matrices is determined by the size of the one-particle-one-hole ( $1ph$ ) space which becomes really huge for deformed and heavy spherical systems.

RPA problem becomes much simpler if the residual two-body interaction of a given multipolarity  $\lambda\mu$  is factorized (reduced to a separable form):

$$\sum_{mnij} \langle mn | V_{res}^{\lambda\mu} | ij \rangle a_m^+ a_n^+ a_j a_i \rightarrow \sum_{k,k'=1}^K \kappa_{k,k'}^{\lambda\mu} \hat{X}_k^{\lambda\mu} \hat{X}_{k'}^{\lambda\mu},$$

$$\hat{X}_k^{\lambda\mu} = \sum_{ph} \langle p | \hat{X}_k^{\lambda\mu} | h \rangle a_p^+ a_h \quad (1)$$

where  $\hat{X}_k^{\lambda\mu}$  are hermitian one-body operators and  $\kappa_{k,k'}^{\lambda\mu}$  are strength constants. The factorization allows to reduce a high-rank RPA matrix to a small matrix with a rank determined by the number of the separable terms. The main problem is to accomplish the factorization self-consistently, with minimal number of separable terms and with high accuracy.

Several self-consistent schemes [2, 3, 4, 5, 6, 7, 8, 8, 9] were proposed during last decades and signified a certain progress in this problem. However, these schemes are not sufficiently general. Some of them are limited to analytical or simple numerical estimates [2, 3, 4, 5], the others are not fully self-consistent [7]. Quite promising is the approach [8, 9] for Skyrme forces. However, it still deals with RPA matrices of rather high rank ( $\sim 400$ ). Besides, it neglects contributions to the residual forces from the Coulomb interaction and time-odd densities and currents.

In this connection, we proposed some time ago a general self-consistent separable RPA (SRPA) approach relevant to arbitrary density- and current-dependent functionals [10, 11, 12, 13]. The method was implemented to the Skyrme functional [14, 15]. In SRPA the one-body operators and strength constants of the separable expansion are unambiguously derived

from the given energy functional. Since we use the self-consistent procedure, there is not need in any new parameters in addition to those in the initial functional. The factorization dramatically reduces rank the of RPA matrix. Usually a few separable terms (or even one term) are enough for accurate reproduction of the genuine residual interaction  $V_{res}^{\lambda\mu}$  [10]. Such impressive result becomes possible due to the effective self-consistent procedure [4] based on solid physical arguments. Besides, the separable operators are constructed to have maxima at different slices of the nucleus and thus cover both nuclear surface and interior [10]. Hence SRPA exhibits accuracy of most involved RPA versions but at much less expense.

One of the main SRPA advantages is its simple and transparent formalism which makes the method very convenient for the analysis and handling of the numerical results. Being self-consistent, SRPA allows to identify spurious admixtures connected with violation of the translational or rotational invariance.

SRPA exploits the full 1ph space and thus equally well treats collective and non-collective states. However, quite often (e.g. for giant resonances) we do not need the detailed RPA description. Then the strength function method, when we completely avoid the calculation of the RPA states, is much more optimal. The separable character of SRPA allows to construct the strength function based on the Lorentz smoothing function. The strength function is naturally separated into two terms, from the mean-field and correlations.

SRPA has a peculiarity to incorporates to the residual interaction the contributions of both time-even and time-odd variables (densities and currents). The time-odd variables naturally appear in the Skyrme functional constructed to contain all the possible bilinear forms from the basic nucleon and spin densities together with their derivatives up to the second order [16]. Besides, time-odd densities and currents are necessary to keep the Galilean and gauge invariance of the Skyrme functional [15, 16]. Skyrme functionals of this kind are actively used for investigation of both ground state and dynamics of atomic nuclei (see e.g. [17] and references therein).

The recent studies with Gogny forces show that contributions of the spin-orbital and Coulomb forces to the residual interaction can be important for the description of low-lying states and giant resonances in exotic nuclei [18]. SRPA takes into account both these contributions. We plan to scrutinize their impact in our further studies.

SRPA is quite general and in principle can be applied to a variety of finite Fermi systems and different functionals. For example, it was derived for the Kohn-Sham functional [19, 20]

and widely used for description of linear dynamics of valence electrons in spherical and deformed atomic clusters [21, 22, 23, 24, 25, 26].

In the previous SRPA studies, we considered nuclear dynamics in spherical nuclei [10, 12]. However, ability of SRPA to reduce the computational effort is much more decisive for deformed systems with its huge 1ph configuration space. So, in the present paper we specify the Skyrme SRPA formalism to *axial* atomic nuclei. The pairing and its contribution to the residual interaction are taken into account. One of the aims of this paper is to present the SRPA by systematic and even tutorial way, with all necessary details.

The paper is organized as follows. In Section II, the derivation of the general SRPA formalism is given and discussed. The method is SRPA is specified for Skyrme forces in Sec. III. The choice of the initial operators is discussed in Sec. IV. The summary is done in Sec. V. Details of the formalism can be found in Appendices A-H.

## II. BASIC SRPA EQUATIONS

### A. Main requirements

The present model provides the self-consistent factorization of the residual interaction to the explicit form

$$\hat{V}_{\text{res}} \rightarrow \hat{V}_{\text{res}}^{\text{sep}} = -\frac{1}{2} \sum_{ss'} \sum_{k,k'=1}^K \{ \kappa_{sk,s'k'} \hat{X}_{sk} \hat{X}_{s'k'} + \eta_{sk,s'k'} \hat{Y}_{sk} \hat{Y}_{s'k'} \}. \quad (2)$$

We assume the residual interaction of a fixed multipolarity ( $\lambda$  for spherical nuclei and  $\lambda\mu$  for deformed nuclei) but, for simplicity, skip hereafter the multipole index. In (2), indices  $s$  and  $s'$  label neutrons and protons;  $\hat{X}_{sk}$  and  $\hat{Y}_{sk}$  are time-even and time-odd hermitian one-body operators. Their time-parity properties formally read

$$T \hat{X}_{sk} T^{-1} = \gamma_T^X \hat{X}_{sk}, \quad \gamma_T^X = +1,$$

$$T \hat{Y}_{sk} T^{-1} = \gamma_T^Y \hat{Y}_{sk}, \quad \gamma_T^Y = -1,$$

where  $T$  is the operator of time inversion. The expansion (2) includes time-odd operators because some Skyrme functionals possess both time-even and time-odd densities and currents [1], see their list in the Appendix C. As was mentioned above, the time-odd densities are necessary to keep the Galilean and gauge invariance of the Skyrme functional [15, 16].

Though these densities do not contribute to the *static* mean field Hamiltonian of spin-saturated systems, they can provide time-dependent perturbations and thus have to be taken into account in the description of nuclear dynamics.

The presence of both time-even and time-odd variables naturally leads to formulation of the model in terms of hermitian operators with given time-parity. These operators have the useful property that

$$\langle 0 | [\hat{A}, \hat{B}] | 0 \rangle \sim (1 - \gamma_T^A \gamma_T^B) \quad (3)$$

i.e. the average commutator does not vanish only for operators  $\hat{A}$  and  $\hat{B}$  having the opposite T-parities ( $\gamma_T^A = -\gamma_T^B$ ), see Appendix B 2 for more details. This property will be widely used in our derivation.

The model should satisfy some principle requirements. The expansion (2) has to be self-consistent. It should involve the minimal number of the separable terms and, at the same time, accurately reproduce the true residual interaction. The operators  $\hat{X}_k$  and  $\hat{Y}_k$  and their weights should have simple and physically transparent structure. Below we will develop the scheme which fulfills these requirements.

## B. Time-dependent Hamiltonian

The nucleus is assumed to undergo small-amplitude harmonic vibrations around Hartree-Fock (HF) or Hartree-Fock-Bogoliubov (HFB) ground state. We start with a general time-dependent functional  $E(J_s^\alpha(\vec{r}, t))$  involving a set of arbitrary neutron and proton densities and currents  $J_s^\alpha(\vec{r}, t)$  (where  $s = n, p$ ;  $\alpha$  labels densities and currents)

$$E(J_s^\alpha(\vec{r}, t)) = \langle \Psi(t) | \hat{H} | \Psi(t) \rangle = \int \mathcal{H}(\vec{r}) d\vec{r} \quad (4)$$

where  $|\Psi(t)\rangle$  is the wave function of the vibrating system described as the time-dependent Slater determinant. Time-dependent densities and currents are determined through the corresponding operators as

$$J_s^\alpha(\vec{r}, t) = {}_s \langle \Psi(t) | \hat{J}_s^\alpha(\vec{r}) | \Psi(t) \rangle = \sum_{h \in s}^{occ} \varphi_h^*(\vec{r}, t) \hat{J}_\alpha(\vec{r}) \varphi_h(\vec{r}, t) \quad (5)$$

where  $\varphi_h(\vec{r}, t)$  is wave function of the hole (occupied) single-particle state. The set (5) includes both time-even and time-odd densities and currents, see examples in the Appendix C.

In the linear regime, the time-dependent density reads as a sum of the static part and the small time-dependent perturbation:

$$J_s^\alpha(\vec{r}, t) = \bar{J}_s^\alpha(\vec{r}) + \delta J_s^\alpha(\vec{r}, t) \quad (6)$$

Then, substituting (6) into (4) and keeping the terms up to the linear order for  $\delta J_s^\alpha(\vec{r}, t)$ , one gets the single-particle Hamiltonian

$$\hat{h}(t) = \hat{h}_0 + \hat{h}_{\text{res}}(t) \quad (7)$$

with the static mean-field part

$$\hat{h}_0 = \sum_{\alpha, s} \frac{\delta E(J_n^\alpha, J_p^\alpha)}{\delta J_s^\alpha} \hat{J}_s^\alpha \quad (8)$$

and the time-dependent response

$$\hat{h}_{\text{res}}(t) = \sum_{\alpha' s'} \left[ \frac{\delta \hat{h}_0}{\delta J_{s'}^{\alpha'}} \right]_{J=\bar{J}} \delta J_{s'}^{\alpha'}(t) = \sum_{\alpha s} \sum_{\alpha' s'} \left[ \frac{\delta^2 E(\vec{r})}{\delta J_s^\alpha \delta J_{s'}^{\alpha'}} \right]_{J=\bar{J}} \delta J_{s'}^{\alpha'}(t) \hat{J}_s^\alpha. \quad (9)$$

The later determines oscillations of the system. For the brevity of notation, we skip the dependence on space coordinates in (7)-(9) and hereafter in this section. The explicit space dependence of the key SRPA operators and values can be found in the Sec. III.

The next step in our derivation is to specify the unknown density variations (or transition densities)

$$\delta J_s^\alpha(t) = \langle \Psi(t) | \hat{J}_s^\alpha | \Psi(t) \rangle - \langle 0 | \hat{J}_s^\alpha | 0 \rangle \quad (10)$$

where  $|0\rangle$  is the static ground state. For this aim we should define the perturbed many-body wave function  $|\Psi(t)\rangle$ .

### C. Scaling perturbed wave function

The macroscopic perturbed many-body wave function  $|\Psi(t)\rangle_s$  is obtained from the static HF or HFB ground state  $|0\rangle_s$  by the scaling transformation [4]

$$|\Psi(t)\rangle_s = \prod_{k=1}^K \exp[-iq_{sk}(t)\hat{P}_{sk}] \exp[-ip_{sk}(t)\hat{Q}_{sk}] |0\rangle_s. \quad (11)$$

Here both  $|\Psi(t)\rangle_s$   $|0\rangle_s$  are Slater determinants;  $\hat{Q}_{sk}(\vec{r})$  and  $\hat{P}_{sk}(\vec{r})$  are generalized coordinate (time-even) and momentum (time-odd) hermitian operators with the properties

$$\begin{aligned}\hat{Q}_{sk} &= \hat{Q}_{sk}^+, & \gamma_T^Q &= 1, \\ \hat{P}_{sk} &= i[\hat{H}, \hat{Q}_{sk}]_{ph} = \hat{P}_{sk}^+, & \gamma_T^P &= -1\end{aligned}\quad (12)$$

where  $\hat{H} = \hat{h}_0 + \hat{V}_{\text{res}}$  is the full Hamiltonian embracing both one-body and two-body parts. The subscript  $ph$  in the commutator means the mapping into particle-hole domain. If the functional includes only time-even densities, then  $\hat{V}_{\text{res}}$  does not contribute to the commutator and so one may use  $\hat{h}_0$  instead of  $\hat{H}$ .

Operators (12) generate time-even and time-odd real collective deformations  $q_{sk}(t)$  and  $p_{sk}(t)$ . Using (11), the transition densities (10) are expressed through these deformations as

$$\begin{aligned}\delta J_s^\alpha(t) &= \langle \Psi(t) | \hat{J}_s^\alpha | \Psi(t) \rangle - \langle 0 | \hat{J}_s^\alpha | 0 \rangle = \\ &= i \sum_k \{ q_{sk}(t) \langle 0 | [\hat{P}_{sk}, \hat{J}_s^\alpha] | 0 \rangle + p_{sk}(t) \langle 0 | [\hat{Q}_{sk}, \hat{J}_s^\alpha] | 0 \rangle \}.\end{aligned}\quad (13)$$

Then the response Hamiltonian (9) can be recast as

$$\hat{h}_{\text{res}}(t) = \sum_{sk} \{ q_{sk}(t) \hat{X}_{sk} + p_{sk}(t) \hat{Y}_{sk} \} \quad (14)$$

where all time-independent terms are collected in the hermitian one-body operators

$$\hat{X}_{sk} = \sum_{s'} \hat{X}_{sk}^{s'} = \sum_{s'} i \sum_{\alpha'\alpha} \left[ \frac{\delta^2 E}{\delta J_{s'}^{\alpha'} \delta J_s^\alpha} \right]_{J=\bar{J}} \langle 0 | [\hat{P}_{sk}, \hat{J}_s^\alpha] | 0 \rangle \hat{J}_{s'}^{\alpha'}, \quad (15)$$

$$\hat{Y}_{sk} = \sum_{s'} \hat{Y}_{sk}^{s'} = \sum_{s'} i \sum_{\alpha'\alpha} \left[ \frac{\delta^2 E}{\delta J_{s'}^{\alpha'} \delta J_s^\alpha} \right]_{J=\bar{J}} \langle 0 | [\hat{Q}_{sk}, \hat{J}_s^\alpha] | 0 \rangle \hat{J}_{s'}^{\alpha'} \quad (16)$$

with the properties

$$\hat{X}_{sk} = \hat{X}_{sk}^+, \quad \gamma_T^X = +1, \quad \hat{X}^* = \hat{X}, \quad (17)$$

$$\hat{Y}_{sk} = \hat{Y}_{sk}^+, \quad \gamma_T^Y = -1, \quad \hat{Y}^* = -\hat{Y}. \quad (18)$$

As is shown below,  $\hat{X}_{sk}$  and  $\hat{Y}_{sk}$  are just the time-even and time-odd operators to be exploited in the separable expansion (2). Following the property (3), time-even densities contribute only to  $\hat{X}_{sk}$  while time-odd densities only to  $\hat{Y}_{sk}$ . The upper index  $s'$  in the operators (15)-(16) determines the isospin (proton or neutron) subspace where these operators act. This is the domain of the density operator  $\hat{J}_{s'}^{\alpha'}$  entering (15)-(16).

To complete the construction of the separable expansion (2), we should still determine the matrices of the strength constants  $\kappa_{sk,s'k'}$  and  $\eta_{sk,s'k'}$ . This can be done through variations of the basic operators (13):

$$\begin{aligned}\delta\hat{X}_{sk}(t) &\equiv \langle \Psi(t) | \hat{X}_{sk} | \Psi(t) \rangle - \langle 0 | \hat{X}_{sk} | 0 \rangle = \\ &= i \sum_{s'k'} q_{s'k'}(t) \langle 0 | [\hat{P}_{s'k'}, \hat{X}_{sk}^{s'}] | 0 \rangle = - \sum_{s'k'} q_{s'k'}(t) \kappa_{s'k',sk}^{-1},\end{aligned}\quad (19)$$

$$\begin{aligned}\delta\hat{Y}_{sk}(t) &\equiv \langle \Psi(t) | \hat{Y}_{sk} | \Psi(t) \rangle - \langle 0 | \hat{Y}_{sk} | 0 \rangle = \\ &= i \sum_{s'k'} p_{s'k'}(t) \langle 0 | [\hat{Q}_{s'k'}, \hat{Y}_{sk}^{s'}] | 0 \rangle = - \sum_{s'k'} p_{s'k'}(t) \eta_{s'k',sk}^{-1}\end{aligned}\quad (20)$$

where

$$\begin{aligned}\kappa_{s'k',sk}^{-1} &= \kappa_{sk,s'k'}^{-1} = -i \langle 0 | [\hat{P}_{s'k'}, \hat{X}_{sk}^{s'}] | 0 \rangle = \\ &= \int d\vec{r} \sum_{\alpha\alpha'} \left[ \frac{\delta^2 E}{\delta J_{s'}^{\alpha'} \delta J_s^{\alpha}} \right] \langle 0 | [\hat{P}_{sk}, \hat{J}_s^{\alpha}] | 0 \rangle \langle 0 | [\hat{P}_{s'k'}, \hat{J}_{s'}^{\alpha'}] | 0 \rangle,\end{aligned}\quad (21)$$

$$\begin{aligned}\eta_{s'k',sk}^{-1} &= \eta_{sk,s'k'}^{-1} = -i \langle 0 | [\hat{Q}_{s'k'}, \hat{Y}_{sk}^{s'}] | 0 \rangle = \\ &= \int d\vec{r} \sum_{\alpha\alpha'} \left[ \frac{\delta^2 E}{\delta J_{s'}^{\alpha'} \delta J_s^{\alpha}} \right] \langle 0 | [\hat{Q}_{sk}, \hat{J}_s^{\alpha}] | 0 \rangle \langle 0 | [\hat{Q}_{s'k'}, \hat{J}_{s'}^{\alpha'}] | 0 \rangle.\end{aligned}\quad (22)$$

Eqs. (21)-(22) represent elements of the symmetric matrix which is inverse to the matrix of the strength constants in (2). Indeed Eqs. (19)-(20) can be recast to

$$- \sum_{sk} \kappa_{s'k',sk} \delta\hat{X}_{sk}(t) = q_{s'k'}(t), \quad (23)$$

$$- \sum_{sk} \eta_{s'k',sk} \delta\hat{Y}_{sk}(t) = p_{s'k'}(t). \quad (24)$$

Then the response Hamiltonian (14) gains the form

$$\hat{h}_{\text{res}}(t) = - \sum_{s'k'} \sum_{sk} \{ \kappa_{s'k',sk} \delta\hat{X}_{sk}(t) \hat{X}_{s'k'} + \eta_{s'k',sk} \delta\hat{Y}_{sk}(t) \hat{Y}_{s'k'} \} \quad (25)$$

which leads to the same eigenvalue problem as the separable Hamiltonian

$$\hat{H} = \hat{h}_0 + \hat{V}_{\text{res}}^{\text{sep}} \quad (26)$$

with  $\hat{V}_{\text{res}}^{\text{sep}}$  from (2). See also [2] for relevant discussion.



In principle, we have already in our disposal the formalism for linear regime of the collective motion in terms of collective harmonic variables

$$q_{sk}(t) = \bar{q}_{sk} \cos(\omega t) = \frac{1}{2} \bar{q}_{sk} (e^{i\omega t} + e^{-i\omega t}) , \quad (27)$$

$$p_{sk}(t) = \bar{p}_{sk} \sin(\omega t) = \frac{1}{2i} \bar{p}_{sk} (e^{i\omega t} - e^{-i\omega t}) . \quad (28)$$

Indeed, Eqs. (15), (16), (21), and (22) deliver the one-body operators and strength matrices which we need for the separable expansion of the two-body interaction. The substitution of the response Hamiltonian (25) and the perturbed wave function (11) into time-dependent HF equation

$$i \frac{d}{dt} |\Psi(t)\rangle = (\hat{h}_0 + \hat{h}_{\text{res}}(t)) |\Psi(t)\rangle \quad (29)$$

would result in the eigenvalue problem. The number  $K$  of the collective variables (and thus of the separable terms) depends on the accuracy we need in the description of collective modes (see discussion in Sec. IV). For  $K = 1$ , the method reduces to the sum rule approach with one collective mode [27]. For  $K > 1$ , we have a system of  $K$  coupled oscillators and the method is reduced to so-called local RPA [27, 28] suitable for a rough description of main branching and gross-structure properties of collective modes. However, the method is still not ready to describe the Landau fragmentation. For this aim, we should consider the detailed 1ph space. This will be done in the next subsection.

#### D. Coupling with 1ph space

Collective modes can be viewed as superpositions of 1ph configurations. To derive this relation, it is convenient to introduce an alternative perturbed many-body wave function

$$|\Psi(t)\rangle = (1 + \sum_s \sum_{ph \in s} c_{ph}^s(t) \hat{A}_{ph}^+) |0\rangle \quad (30)$$

where

$$\hat{A}_{ph}^+ = a_p^\dagger a_h \quad (31)$$

is operator of the creation of 1ph pair and

$$c_{ph}^s(t) = c_{ph}^{s+} e^{i\omega t} + c_{ph}^{s-} e^{-i\omega t} \quad (32)$$

are time-dependent amplitudes of particle-hole configurations in the perturbed state. Here we used the Thouless theorem [29] which establishes the connection between two arbitrary

Slater determinants. The wave function (30) is obviously the *microscopic* counterpart of the *macroscopic* wave function (11).

Substituting (14) and (30) into the time-dependent HF equation (29), one gets, in the linear approximation, the relation between  $c_{ph}^{\pm}$  and collective deformations  $\bar{q}_{sk}$  and  $\bar{p}_{sk}$

$$c_{ph\epsilon s}^{s\pm} = -\frac{1}{2} \frac{\sum_{s'k'} [\bar{q}_{s'k'} < ph | \hat{X}_{s'k'}^s | 0 > \mp i \bar{p}_{s'k'} < ph | \hat{Y}_{s'k'}^s | 0 >]}{\varepsilon_{ph} \pm \omega}, \quad (33)$$

where  $\varepsilon_{ph}$  is the energy of 1ph pair.

In addition to Eqs. (19)-(20), the variations  $\delta \hat{X}_{sk}(t)$  and  $\delta \hat{Y}_{sk}(t)$  can be now obtained with the microscopic perturbed wave function (30):

$$\delta \hat{X}_{sk}(t) = \sum_{s'} \sum_{ph\epsilon s'} (c_{ph}^{s'}(t))^* < ph | \hat{X}_{sk}^{s'} | 0 > + c_{ph}^{s'}(t) < 0 | \hat{X}_{sk}^{s'} | ph >, \quad (34)$$

$$\delta \hat{Y}_{sk}(t) = \sum_{s'} \sum_{ph\epsilon s'} (c_{ph}^{s'}(t))^* < ph | \hat{Y}_{sk}^{s'} | 0 > + c_{ph}^{s'}(t) < 0 | \hat{Y}_{sk}^{s'} | ph >. \quad (35)$$

### E. Eigenvalue problem

But both amplitudes  $c_{ph}^{s\pm}$  and collective variables  $\bar{q}_{sk}$  and  $\bar{p}_{sk}$  are still unknown. The time-dependent HF equation was already exploited and cannot be used once more to determine these unknowns. Thus we need for this aim some additional physical constraint. It can be naturally formulated as equality of the dynamical variations of the basic operators  $\delta \hat{X}_k$  and  $\delta \hat{Y}_k$ , obtained with the macroscopic (11) and microscopic (30) perturbed wave functions. Thus we should equate (19)-(20) and (34)-(35). This gives

$$-\sum_{s'k'} q_{s'k'}(t) \kappa_{s'k',sk}^{-1} = \sum_{s'} \sum_{ph\epsilon s'} (c_{ph}^{s'}(t))^* < ph | \hat{X}_{sk}^{s'} | 0 > + c_{ph}^{s'}(t) < 0 | \hat{X}_{sk}^{s'} | ph >, \quad (36)$$

$$-\sum_{s'k'} q_{s'k'}(t) \eta_{s'k',sk}^{-1} = \sum_{s'} \sum_{ph\epsilon s'} (c_{ph}^{s'}(t))^* < ph | \hat{Y}_{sk}^{s'} | 0 > + c_{ph}^{s'}(t) < 0 | \hat{Y}_{sk}^{s'} | ph >. \quad (37)$$

Substituting (28) and (32) into these expressions and collecting, for example, the terms with  $e^{i\omega t}$ , we gets

$$-\sum_{s'k'} \bar{q}_{s'k'} \kappa_{s'k',sk}^{-1} = 2 \sum_{s'} \sum_{ph\epsilon s'} \{ (c_{ph}^{s'-})^* < ph | \hat{X}_{sk}^{s'} | 0 > + c_{ph}^{s'+} < 0 | \hat{X}_{sk}^{s'} | ph > \}, \quad (38)$$

$$i \sum_{s'k'} \bar{p}_{s'k'} \eta_{s'k',sk}^{-1} = 2 \sum_{s'} \sum_{ph\epsilon s'} \{ (c_{ph}^{s'-})^* < ph | \hat{Y}_{sk}^{s'} | 0 > + c_{ph}^{s'+} < 0 | \hat{Y}_{sk}^{s'} | ph > \}. \quad (39)$$

Then, by using (33), all the unknowns in the expressions are reduced to the collective variables  $\bar{q}_k$  and  $\bar{p}_k$  and we finally get

$$\begin{aligned} \sum_{\bar{s}\bar{k}} \{ \bar{q}_{\bar{s}\bar{k}} [F_{s'k',\bar{s}\bar{k}}^{(XX)} - \kappa_{\bar{s}\bar{k},s'k'}^{-1}] + \bar{p}_{\bar{s}\bar{k}} F_{s'k',\bar{s}\bar{k}}^{(XY)} \} &= 0, \\ \sum_{\bar{s}\bar{k}} \{ \bar{q}_{\bar{s}\bar{k}} F_{s'k',\bar{s}\bar{k}}^{(YX)} + \bar{p}_{\bar{s}\bar{k}} [F_{s'k',\bar{s}\bar{k}}^{(YY)} - \eta_{\bar{s}\bar{k},s'k'}^{-1}] \} &= 0 \end{aligned} \quad (40)$$

with

$$\begin{aligned} F_{s'k',\bar{s}\bar{k}}^{(XX)} &= \sum_s \sum_{ph \in s} \frac{1}{\varepsilon_{ph}^2 - \omega^2} \{ \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle^* \langle ph | \hat{X}_{s'k'}^s | 0 \rangle (\varepsilon_{ph} + \omega) \\ &\quad + \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle \langle 0 | \hat{X}_{s'k'}^s | ph \rangle (\varepsilon_{ph} - \omega) \}, \end{aligned} \quad (41)$$

$$\begin{aligned} F_{s'k',\bar{s}\bar{k}}^{(YX)} &= -i \sum_s \sum_{ph \in s} \frac{1}{\varepsilon_{ph}^2 - \omega^2} \{ \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle^* \langle ph | \hat{X}_{s'k'}^s | 0 \rangle (\varepsilon_{ph} + \omega) \\ &\quad + \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle \langle 0 | \hat{X}_{s'k'}^s | ph \rangle (\varepsilon_{ph} - \omega) \}, \end{aligned} \quad (42)$$

$$\begin{aligned} F_{s'k',\bar{s}\bar{k}}^{(XY)} &= i \sum_s \sum_{ph \in s} \frac{1}{\varepsilon_{ph}^2 - \omega^2} \{ \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle^* \langle ph | \hat{Y}_{s'k'}^s | 0 \rangle (\varepsilon_{ph} + \omega) \\ &\quad + \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | h \rangle \langle 0 | \hat{Y}_{s'k'}^s | ph \rangle (\varepsilon_{ph} - \omega) \}, \end{aligned} \quad (43)$$

$$\begin{aligned} F_{s'k',\bar{s}\bar{k}}^{(YY)} &= \sum_s \sum_{ph \in s} \frac{1}{\varepsilon_{ph}^2 - \omega^2} \{ \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle^* \langle ph | \hat{Y}_{s'k'}^s | 0 \rangle (\varepsilon_{ph} + \omega) \\ &\quad + \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle \langle 0 | \hat{Y}_{s'k'}^s | ph \rangle (\varepsilon_{ph} - \omega) \}. \end{aligned} \quad (44)$$

These equations can be simplified using the relations

$$\langle 0 | \hat{X}_{\bar{s}\bar{k}}^s | ph \rangle = \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle^* = \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle = \overline{\langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle}, \quad (45)$$

$$\langle 0 | \hat{Y}_{\bar{s}\bar{k}}^s | ph \rangle = \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle^* = - \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle = -i \overline{\langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle} \quad (46)$$

which directly follow from the properties (17) and (18) and expressions (15) and (16). Then elements of the RPA matrix can be rewritten in terms of real (overline) matrix elements as

$$F_{s'k',\bar{s}\bar{k}}^{(XX)} = 2 \sum_s \sum_{ph \in s} \frac{\overline{\varepsilon_{ph} \langle ph | \hat{X}_{\bar{s}\bar{k}}^s | 0 \rangle} \overline{\langle ph | \hat{X}_{s'k'}^s | 0 \rangle}}{\varepsilon_{ph}^2 - \omega^2}, \quad (47)$$

$$F_{s'k',\bar{s}\bar{k}}^{(YX)} = F_{s'k',\bar{s}\bar{k}}^{(XY)} = -2 \sum_s \sum_{ph \in s} \frac{\overline{\omega \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle} \overline{\langle ph | \hat{X}_{s'k'}^s | 0 \rangle}}{\varepsilon_{ph}^2 - \omega^2}, \quad (48)$$

$$F_{s'k',\bar{s}\bar{k}}^{(YY)} = 2 \sum_s \sum_{ph \in s} \frac{\overline{\varepsilon_{ph} \langle ph | \hat{Y}_{\bar{s}\bar{k}}^s | 0 \rangle} \overline{\langle ph | \hat{Y}_{s'k'}^s | 0 \rangle}}{\varepsilon_{ph}^2 - \omega^2}. \quad (49)$$

Supposing determinant of the system (40) to be zero, we obtain the dispersion equation for determination of the RPA eigenvalues  $\omega_\nu$ .

## F. Normalization condition

The standard RPA operators of excited one-phonon states  $|\nu\rangle$  are defined as

$$\hat{Q}_\nu^+ = \frac{1}{2} \sum_s \sum_{ph \in s} \{c_{ph}^{\nu s-} \hat{A}_{ph}^+ - c_{ph}^{\nu s+} \hat{A}_{ph}\} \quad (50)$$

and fulfill the normalization and orthogonality conditions

$$\langle [\hat{Q}_\nu, \hat{Q}_{\nu'}^+] \rangle = \delta_{\nu, \nu'}, \quad \langle [\hat{Q}_\nu^+, \hat{Q}_{\nu'}^+] \rangle = \langle [\hat{Q}_\nu, \hat{Q}_{\nu'}] \rangle = 0, \quad (51)$$

where  $\hat{A}_{ph}^+$  and  $c_{ph}^{\nu \pm}$  are given by (31) and (33), respectively. In the quasi-boson approximation ( $[\hat{A}_{ph}, \hat{A}_{p'h'}^+] = \delta_{pp'} \delta_{hh'}$ ), the normalization condition  $[\hat{Q}_\nu, \hat{Q}_\nu^+] = 1$  results in the relation

$$\sum_s \sum_{ph \in s} \{(c_{ph}^{\nu s-})^2 - (c_{ph}^{\nu s+})^2\} = 2. \quad (52)$$

By using (33), it can be recast in terms of the RPA matrix coefficients (47)-(49):

$$\begin{aligned} & \sum_s \sum_{ph \in s} \{(c_{ph}^{\nu s-})^2 - (c_{ph}^{\nu s+})^2\} \\ &= \sum_{s'k'} \sum_{\bar{s}\bar{k}} \frac{1}{4} \left\{ \bar{q}_{s'k'}^\nu \bar{q}_{\bar{s}\bar{k}}^\nu \frac{\partial F_{s'k', \bar{s}\bar{k}}^{(XX)}(\omega_\nu)}{\partial \omega_\nu} + 2 \bar{q}_{s'k'}^\nu \bar{p}_{\bar{s}\bar{k}}^\nu \frac{\partial F_{s'k', \bar{s}\bar{k}}^{(YX)}(\omega_\nu)}{\partial \omega_\nu} + \bar{p}_{s'k'}^\nu \bar{p}_{\bar{s}\bar{k}}^\nu \frac{\partial F_{s'k', \bar{s}\bar{k}}^{(YY)}(\omega_\nu)}{\partial \omega_\nu} \right\} = 2N_\nu. \end{aligned} \quad (53)$$

Hence the variables  $\bar{q}_{sk}^\nu$  and  $\bar{p}_{sk}^\nu$  should be renormalized by the factor  $1/\sqrt{N_\nu}$ .

## G. Pairing contribution

The pairing Hamiltonian reads

$$\hat{h}_{\text{pair}} = - \sum_{s=n,p} G_s \hat{\chi}_s^+ \hat{\chi}_s \quad (54)$$

where

$$\hat{\chi}_s^+ = \sum_{j \in s} a_j^+ a_j^+, \quad \hat{\chi}_s = \sum_{j \in s} a_j a_j. \quad (55)$$

Then, after the Bogoliubov transformation from particle to quasiparticle operators, the hermitian one-body operators have a general form

$$\hat{A} = \sum_{ij} \langle ij|A|0 \rangle (a_i^+ a_j + a_j^+ a_i) \quad (56)$$

$$\begin{aligned} &= 2 \sum_{ij}^{K_j, K_i > 0} \{ \langle ij|A|0 \rangle (u_i v_j + \gamma_T^A u_j v_i) (\hat{\alpha}_i^+ \hat{\alpha}_j^+ + \gamma_T^A \hat{\alpha}_i \hat{\alpha}_j) \\ &\quad + (u_i u_j - \gamma_T^A v_j v_i) (\hat{\alpha}_j^+ \hat{\alpha}_i - \gamma_T^A \hat{\alpha}_i \hat{\alpha}_j^+) \} \end{aligned} \quad (57)$$

where  $\bar{i}$  are time-reversed states and the time-inverse factor  $\gamma_T^A$  defines time-parity of the operator  $\hat{A}$  (see details in Appendix B1).

Hence the time-even ( $\gamma_T^A = 1$ ) and time-odd ( $\gamma_T^B = -1$ ) operators read

$$\hat{A} = 2 \sum_{ij}^{K_j, K_i > 0} \langle ij|A|0 \rangle (u_i v_j + u_j v_i) (\hat{\alpha}_i^+ \hat{\alpha}_j^+ + \hat{\alpha}_{\bar{i}} \hat{\alpha}_j), \quad (58)$$

$$\hat{B} = 2 \sum_{ij}^{K_j, K_i > 0} \langle ij|B|0 \rangle (u_i v_j - u_j v_i) (\hat{\alpha}_i^+ \hat{\alpha}_j^+ - \hat{\alpha}_{\bar{i}} \hat{\alpha}_j), \quad (59)$$

i.e. obtain the pairing factors

$$u_{ij}^{(+)} = u_i v_j + u_j v_i, \quad u_{ij}^{(-)} = u_i v_j - u_j v_i. \quad (60)$$

This is the case of time even-operators  $\hat{Q}_{sk}$  and  $\hat{X}_{sk}$  and the time-odd operator  $\hat{Y}_{sk}$ . The situation with the time-odd operator

$$\hat{P}_{sk} = i[\hat{H}, \hat{Q}_{sk}] = i\{\hat{h}_0, \hat{Q}_{sk}\} + [\hat{V}_{\text{res}}^{\text{sep}}, \hat{Q}_{sk}] = i[\hat{h}_0, \hat{Q}_{sk}] - \hat{Y}_{sk}$$

is more complicated because of the additional term  $i[\hat{h}_0, \hat{Q}_{sk}]$ . Taking into account (58) and (??) this operator reads

$$\hat{P}_{sk} = 2 \sum_{ij \in s}^{K_j, K_i > 0} \{i2\epsilon_{ij} u_{ij}^{(+)} \langle ij|Q_{sk}|0 \rangle - u_{ij}^{(-)} \langle ij|Y_{sk}^s|0 \rangle\} (\hat{\alpha}_i^+ \hat{\alpha}_j^+ - \hat{\alpha}_{\bar{i}} \hat{\alpha}_j). \quad (61)$$

It is seen that  $\hat{P}_{sk}$  keeps the same operator structure  $\hat{\alpha}_i^+ \hat{\alpha}_j^+ - \hat{\alpha}_{\bar{i}} \hat{\alpha}_j$  as (59) but, at the same has, unlike  $\hat{Y}_{sk}$ , the diagonal ( $i = j$  matrix elements. The later is the obvious consequence of the term  $i[\hat{h}_0, \hat{Q}_{sk}]$ . Explicit expressions for the matrix elements of the operator  $\hat{P}_{sk}$  are given in the Appendix G1.

Besides the factors (60) the pairing results in the specific contribution to the time-even part of the response Hamiltonian  $\hat{h}_{\text{res}}(t)$ . This contribution can be derived in close analogy with other time-even densities. Then we get

$$\delta \hat{h}_{\text{res}}^{\text{pair}}(t) = \sum_{ks} q_{ks}(t) \sum_s \hat{X}_{ks}^{s(\text{pair})} \quad (62)$$

where

$$\hat{X}_{ks}^{s(\text{pair})} = -G_s \chi_{X,sk} \sum_{j \in s} (u_j^2 - v_j^2) (\alpha_{\bar{j}} \alpha_j + \alpha_j^+ \alpha_j^+). \quad (63)$$

The pairing response reads

$$\begin{aligned}
\chi_{X,sk} &= i \langle \tilde{0} | [\hat{P}_{ks}, \hat{\chi}_s^+ + \hat{\chi}_s] | \tilde{0} \rangle \\
&= i \sum_{j \in s} 8i \epsilon_j u_{jj}^{(+)} \langle j\bar{j} | Q_{ks} | 0 \rangle \sum_{l \in s} (u_l^2 - v_l^2) \langle \tilde{0} | [(\alpha_j^+ \alpha_j^+ - \alpha_j^- \alpha_j^-), (\alpha_l^- \alpha_l^- + \alpha_l^+ \alpha_l^+)] | \tilde{0} \rangle \\
&= 16 \sum_{j \in s} \epsilon_j (u_j^2 - v_j^2) u_{jj}^{(+)} \langle j\bar{j} | Q_{ks} | 0 \rangle
\end{aligned} \tag{64}$$

where we used the diagonal part of the operator (61). Finally, the pairing contribution to the strength matrix is

$$[\kappa_{sk'sk'}^{(pair)}]^{-1} = -i \langle \tilde{0} | [\hat{P}_{k's}, \hat{X}_{sk}^{s(pair)}] | \tilde{0} \rangle = G_s \chi_{X,sk'} \chi_{X,sk}. \tag{65}$$

It worth noting that in all the sections except of the present one, the pairing factors (60) are supposed to be included into the matrix elements and transition densities and thus are not depicted explicitly.

## H. Strength function method

In exploration of the system response to external fields, we are usually interested in the total strength function instead of the responses of particular RPA states. For example, giant resonances in heavy nuclei are formed by thousands of RPA states whose contributions in any case cannot be distinguished experimentally. In this case, it is reasonable to implement the strength function formalism. Besides, the calculation of the strength function is much easier.

For electric external fields of multipolarity  $E\lambda\mu$ , the strength function can be defined as

$$S_L(E\lambda\mu; \omega) = \sum_{\nu} \omega_{\nu}^L M_{\lambda\mu\nu}^2 \zeta(\omega - \omega_{\nu}) \tag{66}$$

where

$$\zeta(\omega - \omega_{\nu}) = \frac{1}{2\pi} \frac{\Delta}{(\omega - \omega_{\nu})^2 + (\Delta/2)^2} \tag{67}$$

is Lorentz weight with an averaging parameter  $\Delta$  and

$$M_{\lambda\mu\nu} = \frac{1}{\sqrt{2}} \sum_s e_s^{eff} \sum_{ij \in s} \langle ij | \hat{f}_{\lambda\mu} | 0 \rangle (c_{ij}^{\nu s-} + c_{ij}^{\nu s+}) \tag{68}$$

is the matrix element of  $E\lambda\mu$  transition from the ground state to the RPA state  $|\nu\rangle$ . The pairing factor is included to the matrix elements. The forward and backward two-quasiparticle amplitudes  $c_{ij}^{\nu s\pm}$  follow from (33) (with the subsequent normalization). The operator of the electric external field in the long-wave approximation reads

$$\hat{f}_{\lambda\mu} = e \frac{1}{1 + \delta_{\mu,0}} r^\lambda (Y_{\lambda\mu} + Y_{\lambda\mu}^\dagger). \quad (69)$$

Further,  $e_s^{eff}$  is the effective charge (in the dipole case  $e_p^{eff} = N/A$  and  $e_n^{eff} = -Z/A$  and  $e$  is the proton charge);  $\omega_\nu$  is the energy of the the RPA state  $|\nu\rangle$ .

It is worth noting that, unlike the standard definition of the strength function with using  $\delta(\omega - \omega_\nu)$ , we exploit here the Lorentz weight. It is convenient to simulate smoothing effects.

For direct use of the expression (66), we still have to know all the RPA eigenvalues and eigenvectors. This needs an appreciable computational effort. To avoid this, let's recast (66) to the form which does not need the information on the particular RPA states [30]. For this aim, we will use the Cauchy residue theorem. Namely, the strength function will be recast as a sum of the residues for the poles  $z = \pm\omega_\nu$ . Since the sum of all the residues (covering all the poles) is zero, the residues with  $z = \pm\omega_\nu$  (whose calculation is time consuming) can be replaced by the sum of residues with  $z = \omega \pm i(\Delta/2)$  and  $z = \pm\varepsilon_{ph}$  whose calculation is much less expensive. Now let's consider this procedure step by step.

First, we use (33) and rewrite the matrix element (68) to the form

$$M_{E\lambda\mu\nu} = -\frac{1}{\sqrt{2N_\nu}} \sum_s e_s^{eff} \sum_{ij\epsilon s} \frac{\langle ij|f_{\lambda\mu}|0\rangle}{\varepsilon_{ij}^2 - \omega_\nu^2} \quad (70)$$

$$\cdot \sum_{s'k'} \{ \varepsilon_{ij} \bar{q}_{s'k'}^\nu \langle ij|\hat{X}_{s'k'}^s|0\rangle + i\omega_\nu \bar{p}_{s'k'}^\nu \langle ij|\hat{Y}_{s'k'}^s|0\rangle \} \\ = -\frac{1}{\sqrt{2N_\nu}} \sum_{s'k'} \{ \bar{q}_{s'k'}^\nu A_{s'k'}^{(X)}(E\lambda\mu) + \bar{p}_{s'k'}^\nu A_{s'k'}^{(Y)}(E\lambda\mu) \} \quad (71)$$

where

$$A_{s'k'}^{(X)}(E\lambda\mu) = \sum_s e_s^{eff} \sum_{ij\epsilon s} \frac{\varepsilon_{ij} \langle ij|X_{s'k'}^s|0\rangle \langle ij|f_{\lambda\mu}|0\rangle}{\varepsilon_{ij}^2 - \omega_\nu^2}, \quad (72)$$

$$A_{s'k'}^{(Y)}(E\lambda\mu) = i \sum_s e_s^{eff} \sum_{ij\epsilon s} \frac{\omega_\nu \langle ij|Y_{s'k'}^s|0\rangle \langle ij|f_{\lambda\mu}|0\rangle}{\varepsilon_{ij}^2 - \omega_\nu^2}. \quad (73)$$

For the sake of brevity, we introduce the new index  $\beta = \{skg\}$  where  $g = 1$  and  $2$  for time-even and time-odd quantities, respectively. Then the squared matrix element can be

written in the compact form [30, 31]

$$M_{E\lambda\mu}^2 = \frac{1}{2N} \sum_{\beta\beta'} R_\beta R_{\beta'} A_\beta A_{\beta'} = \frac{1}{2N} \sum_{\beta\beta'} 2N \frac{F_{\beta\beta'}}{\frac{\partial}{\partial\omega} \det D} A_\beta A_{\beta'} = \sum_{\beta\beta'} \frac{F_{\beta\beta'}}{\frac{\partial}{\partial\omega} \det F} A_\beta A_{\beta'} \quad (74)$$

where  $F$  is the determinant of RPA matrix (40),  $F_{\beta\beta'}$  is its algebraic supplement and

$$R_{sk\ g=1} = \bar{q}_{sk}^\nu, \quad R_{sk\ g=2} = \bar{p}_{sk}^\nu, \quad (75)$$

$$A_{sk\ g=1} = A_{sk}^{(X)}(E\lambda\mu), \quad A_{sk\ g=2} = A_{sk}^{(Y)}(E\lambda\mu). \quad (76)$$

Substituting (74) to (66), one gets

$$S_L(E\lambda\mu; \omega) = \sum_{\nu} \omega_{\nu}^L \frac{\sum_{\beta\beta'} F_{\beta\beta'}^{\nu} A_{\beta}^{\nu} A_{\beta'}^{\nu}}{\frac{\partial}{\partial\omega_{\nu}} \det F^{\nu}} \zeta(\omega - \omega_{\nu}). \quad (77)$$

Then, taking into account that the determinant  $\det F^{\nu}$  has first-order poles  $\omega = \omega_{\nu}$ , the strength function can be rewritten through the residue of these poles on the complex plain, or equivalently, through the corresponding contour integrals:

$$\begin{aligned} S_L(E\lambda\mu; \omega) &= \sum_{\nu} \text{Res}\left\{ z^L \frac{\sum_{\beta\beta'} F_{\beta\beta'}(z) A_{\beta}(z) A_{\beta'}(z)}{\det F(z)} \zeta(\omega - z) \right\}_{z=\omega_{\nu}} \\ &= \frac{1}{2\pi i} \sum_{\nu} \oint_{z=\omega_{\nu}} z^L \frac{\sum_{\beta\beta'} F_{\beta\beta'}(z) A_{\beta}(z) A_{\beta'}(z)}{\det F(z)} \zeta(\omega - z). \end{aligned} \quad (78)$$

Unlike (77), the denominator in (78) includes the RPA determinant instead of its derivative.

Following Cauchy theorem, sum of all the residues (covering all possible poles of the strength function) is zero and so one can express the residues with  $z = \omega_{\nu}$  through the rest of the others:

$$\text{Res}[S]_{z=\omega_{\nu}} = -(\text{Res}[S]_{z=-\omega_{\nu}} + \text{Res}[S]_{z\rightarrow\infty} + \text{Res}[S]_{z=\omega\pm i(\Delta/2)} + \text{Res}[S]_{z=\pm\varepsilon_{ij}}) \quad (79)$$

where the poles  $z = \omega \pm i(\Delta/2)$  and  $z = \pm\varepsilon_{ij}$  originate from the Lorentz weight and denominator  $F_{\beta\beta'}(z) A_{\beta}(z) A_{\beta'}(z)$ , respectively. The RPA determinant  $F(z)$  has zeros only at  $z = \pm\omega_{\nu}$ .

It's easy to prove that for  $L = 0, 1, 2$  we have  $\lim_{|z|\rightarrow\infty} S(\omega, z) = 0$ . Also,  $\text{Res}[S]_{z=-\omega_{\nu}}$  and  $\text{Res}[S]_{z=-\varepsilon_{ij}}$  can be neglected for large positive  $z$ -values (high energies of giant resonances) and relevant values of the averaging parameter  $\Delta$ . Remaining residues over the poles  $z = \omega \pm i(\Delta/2)$  and  $z = \varepsilon_{ph}$  give the final outcome

$$\text{Res}[S]_{z=\omega_{\nu}} \simeq -\text{Res}[S]_{z=\omega\pm i(\Delta/2)} - \text{Res}[S]_{z=\varepsilon_{ij}}. \quad (80)$$



Finally, the strength function for  $L = 0, 1, 2$  reads

$$S_L(E\lambda\mu, \omega) = \frac{1}{\pi} \Im \left[ \frac{z^L \sum_{\beta\beta'} F_{\beta\beta'}(z) A_\beta(z) A_{\beta'}(z)}{F(z)} \right]_{z=\omega+i(\Delta/2)} \quad (81)$$

$$+ \sum_s (e_s^{eff})^2 \sum_{ij \in s} \varepsilon_{ij}^L \langle ij | f_{\lambda\mu} | 0 \rangle^2 \zeta(\omega - \varepsilon_{ij}).$$

The first term in (81) is contributions of the residual interaction. It vanishes at  $V_{\text{res}} = 0$ . The second term is the unperturbed (purely quasiparticle) strength function.

## I. General discussion

Equations (15)-(16), (21)-(22), (33), (40), (47)-(49), and (53) constitute the basic SRPA formalism. Before proceeding the specification to Skyrme functional it is worth to comment some essential points of the model.

- One may show (e.g. by using a standard derivation of the matrix RPA) that the separable Hamiltonian (26) with one-phonon states (50) results in the SRPA equations (40)-(49) if to express unknowns  $c_{ph}^{\nu\pm}$  through  $\bar{q}_{\bar{k}}$  and  $\bar{p}_{\bar{k}}$ . Familiar RPA equations for unknowns  $c_{ph}^{\nu\pm}$  require the RPA matrix of a high rank equal to size of the  $1ph$  basis. The separable approximation allows to reformulate the RPA problem in terms of a few unknowns  $\bar{q}_{\bar{k}}$  and  $\bar{p}_{\bar{k}}$  (see relation (33)) and thus to minimize the computational effort. As is seen from (40), the rank of the SRPA matrix is equal to  $4K$  (where  $K$  is the number of the separable operators) and hence is quite low.
- The number of RPA states  $|\nu\rangle$  is equal to the number of the relevant  $1ph$  configurations used in the calculations. In heavy nuclei, this number ranges the interval  $10^3$ - $10^4$ . Every RPA state  $|\nu\rangle$  is characterized by the particular set of the values  $\bar{q}_{sk}^\nu$  and  $\bar{p}_{sk}^\nu$  which, following (33), self-consistently regulate relative contributions of different time-even and time-odd operators of the residual interaction to this state.
- Eqs. (15), (16), (21), (22) relate the basic SRPA values with the starting functional and input operators  $\hat{Q}_{sk}$  and  $\hat{P}_{sk}$  by a simple and physically transparent way. After choosing the initial operators  $\hat{Q}_{sk}$ , all other SRPA values are straightforwardly determined following the steps

$$\hat{Q}_{sk} \Rightarrow \langle 0 | [\hat{Q}_{sk}, \hat{J}_s^\alpha] | 0 \rangle \Rightarrow \hat{Y}_{sk}, \eta_{sk,s'k'}^{-1} \Rightarrow \hat{P}_{sk} \Rightarrow \langle 0 | [\hat{P}_{sk}, \hat{J}_s^\alpha] | 0 \rangle \Rightarrow \hat{X}_{sk}, \kappa_{sk,s'k'}^{-1}. \quad (82)$$

As is discussed in Sec. IV, the proper choice of  $\hat{Q}_{sk}$  is crucial to achieve good convergence of the separable expansion (2) at a minimal number of separable operators.

- SRPA restores the conservation laws (e.g. translational invariance) violated by the static mean field. Indeed, let's assume a symmetry mode with the generator  $\hat{P}_{\text{sym}}$ . Then, to keep the conservation law  $[\hat{H}, \hat{P}_{\text{sym}}] = 0$ , we simply have to include  $\hat{P}_{\text{sym}}$  into the set of the input generators  $\hat{P}_{sk}$  together with its complement  $\hat{Q}_{\text{sym}} = i[\hat{H}, \hat{P}_{\text{sym}}]$ .
- The basic SRPA operators can be expressed via the separable residual interaction (2):

$$\hat{X}_{sk} = -i[\hat{V}_{\text{res}}^{\text{sep}}, \hat{P}_{sk}]_{ph}, \quad \hat{Y}_{sk} = -i[\hat{V}_{\text{res}}^{\text{sep}}, \hat{Q}_{sk}]_{ph} \quad (83)$$

where the index  $ph$  means the  $1ph$  part of the commutator. It is seen that the time-odd operator  $\hat{P}_{sk}$  retains the time-even part of  $V_{\text{res}}^{\text{sep}}$  to build  $\hat{X}_{sk}$ . Vice versa, the commutator with the time-even operator  $\hat{Q}_{sk}$  keeps the time-odd part of  $V_{\text{res}}^{\text{sep}}$  to build  $\hat{Y}_{sk}$ . Equations (83) hints also the relation between the SRPA operators and the true (not separable) residual two-body interaction.

- Some of the SRPA values read as averaged commutators between time-odd and time-even operators. This allows to establish useful relations with other models. For example, (21), (22) and (83) give

$$\kappa_{s'k',sk}^{-1} = -i\langle 0 | [\hat{P}_{s'k'}, \hat{X}_{sk}^s] | 0 \rangle = -\langle 0 | [\hat{P}_{s'k'}, [\hat{V}_{\text{res}}^{\text{sep}}, \hat{P}_{sk}]] | 0 \rangle, \quad (84)$$

$$\eta_{s'k',sk}^{-1} = -i\langle 0 | [\hat{Q}_{s'k'}, \hat{Y}_{sk}^s] | 0 \rangle = -\langle 0 | [\hat{Q}_{s'k'}, [\hat{V}_{\text{res}}^{\text{sep}}, \hat{Q}_{sk}]] | 0 \rangle. \quad (85)$$

Similar double commutators (but with the full Hamiltonian instead of  $\hat{V}_{\text{res}}^{\text{sep}}$ ) correspond to  $m_3$  and  $m_1$  sum rules, respectively, and so represent the spring and inertia parameters [28] in the basis of collective generators  $\hat{Q}_{sk}$  and  $\hat{P}_{sk}$ . This demonstrates the connection of the SRPA with the sum rule approach [32, 33] and local RPA [28].

- The commutator form of the SRPA values allows to represent them through the matrix elements from the operators entering the commutators. Namely, the strength constants and responses gain the form (see Appendix B 3)

$$i < 0 | [\hat{Q}_{sk}, \hat{B}_s] | 0 > = -4 \sum_{ij}^{K_i, K_j > 0} < ij | \hat{Q}_{sk} | 0 > \Im \{ < ij | \hat{B}_s | 0 > \}, \quad (86)$$

$$i < 0 | [\hat{P}_{sk}, \hat{B}_s] | 0 > = -4i \sum_{ij}^{K_i, K_j > 0} < ij | \hat{P}_{sk} | 0 > \Re \{ < ij | \hat{B}_s | 0 > \} \quad (87)$$

where  $\hat{B}$  equals to  $\hat{X}$  for the strength constants and to  $\hat{J}$  (the density operator) for the responses. Since the involved matrix elements are already in our disposal, these forms considerably simplify the calculations. Besides, these forms are convenient for analysis. For example, they allow to determine the conditions of vanishing some response components, see Appendix B 3 for more details.

- In fact, SRPA is the first TDHF iteration with the initial wave function (11). A single iteration is generally not enough to get the complete convergence of TDHF results. However, SRPA calculations demonstrate that high accuracy can be achieved even in this case if to ensure the optimal choice of the input operators  $\hat{Q}_{sk}$  and  $\hat{P}_{sk}$  and keep sufficient amount of the separable terms (see discussion in Sec. IV). In this case, the first iteration already gives quite accurate results.
- SRPA equations are very general and, after simple modifications, can be applied to diverse systems (atomic nuclei, atomic clusters, etc.) described by density and current-dependent functionals, see for the case of atomic clusters Refs. [21, 22, 23, 24, 25, 26]. Even Bose systems can be covered if one redefines the many-body wave function (30) exhibiting the perturbation through the elementary excitations. In this case, the Slater determinant for 1ph excitations should be replaced by a perturbed many-body function in terms of elementary bosonic excitations.

### III. SPECIFICATION FOR SKYRME FUNCTIONAL

#### A. Skyrme functional and its mean field Hamiltonian

We use the Skyrme functional [14] in the particular form [15, 16, 27]

$$E = \int d\vec{r} \left( \mathcal{H}_{\text{kin}} + \mathcal{H}_{\text{Sk}}(\rho_s, \tau_s, \vec{\sigma}_s, \vec{j}_s, \vec{J}_s) + \mathcal{H}_{\text{C}}(\rho_p) \right) - E_{\text{cm}}, \quad (88)$$

where

$$\mathcal{H}_{\text{kin}} = \frac{\hbar^2}{2m} \tau, \quad (89)$$

$$\mathcal{H}_{\text{C}} = \frac{e^2}{2} \int d\vec{r}' \rho_p(\vec{r}) \frac{1}{|\vec{r} - \vec{r}'|} \rho_p(\vec{r}') - \frac{3}{4} e^2 \left( \frac{3}{\pi} \right)^{\frac{1}{3}} [\rho_p(\vec{r})]^{\frac{4}{3}}, \quad (90)$$

$$\mathcal{H}_{\text{Sk}} = \frac{b_0}{2} \rho^2 - \frac{b'_0}{2} \sum_s \rho_s^2 - \frac{b_2}{2} \rho(\Delta\rho) + \frac{b'_2}{2} \sum_s \rho_s(\Delta\rho_s) \quad (91)$$

$$\begin{aligned}
& + \frac{b_3}{3} \rho^{\alpha+2} - \frac{b'_3}{3} \rho^\alpha \sum_s \rho_s^2 \\
& + b_1 (\rho \tau - \vec{j}^2) - b'_1 \sum_s (\rho_s \tau_s - \vec{j}_s^2) \\
& - b_4 \left( \rho (\vec{\nabla} \vec{\mathfrak{S}}) + \vec{\sigma} \cdot (\vec{\nabla} \times \vec{j}) \right) - b'_4 \sum_s \left( \rho_s (\vec{\nabla} \vec{\mathfrak{S}}_s) + \vec{\sigma}_s \cdot (\vec{\nabla} \times \vec{j}_s) \right)
\end{aligned}$$

are kinetic, Coulomb and Skyrme terms respectively. The densities and currents used in this functional are defined in the Appendix C. Densities without the index  $s$  involve both neutrons and protons, e.g.  $\rho = \rho_p + \rho_n$ . Parameters  $b$  and  $\alpha$  are fitted to describe ground state properties of atomic nuclei. The value  $E_{cm}$  is the center mass correction.

First functional derivatives read [27]

$$U_s(\vec{r}) = \frac{\delta E}{\delta \rho_s(\vec{r})} = b_0 \rho(\vec{r}) - b'_0 \rho_s(\vec{r}) \quad (92)$$

$$\begin{aligned}
& + b_3 \frac{\alpha + 2}{3} \rho^{\alpha+1}(\vec{r}) - \frac{b'_3}{3} \{ \alpha \rho^{\alpha-1}(\vec{r}) \sum_{s'} \rho_{s'}^2(\vec{r}) + 2 \rho^\alpha(\vec{r}) \rho_s(\vec{r}) \} \\
& + b_1 \tau(\vec{r}) - b'_1 \tau_s(\vec{r}) - b_2 \Delta_{\vec{r}} \rho(\vec{r}) + b'_2 \Delta_{\vec{r}} \rho_s(\vec{r}) \\
& - b_4 \vec{\nabla}_{\vec{r}} \vec{\mathfrak{S}}(\vec{r}) - b'_4 \vec{\nabla}_{\vec{r}} \vec{\mathfrak{S}}_s(\vec{r}) \\
& + \delta_{s,p} e^2 \{ \int d\vec{r}_1 \frac{\rho_p(\vec{r}_1)}{|\vec{r} - \vec{r}_1|} - (\frac{3}{\pi})^{1/3} [\rho_p(\vec{r})]^{1/3} \},
\end{aligned}$$

$$B_s(\vec{r}) = \frac{\delta E}{\delta \tau_s(\vec{r})} = \frac{\hbar^2}{2m} + b_1 \rho(\vec{r}) - b'_1 \rho_s(\vec{r}), \quad (93)$$

$$\vec{W}_s(\vec{r}) = \frac{\delta E}{\delta \vec{\mathfrak{S}}_s(\vec{r})} = b_4 \vec{\nabla}_{\vec{r}} \rho(\vec{r}) + b'_4 \vec{\nabla}_{\vec{r}} \rho_s(\vec{r}). \quad (94)$$

for time-even densities and currents and

$$\vec{A}_s(\vec{r}) = \frac{\delta E}{\delta \vec{j}_s(\vec{r})} = -2b_1 \vec{j}(\vec{r}) + 2b'_1 \vec{j}_s(\vec{r}) - b_4 (\vec{\nabla}_{\vec{r}} \times \vec{\sigma}(\vec{r})) - b'_4 (\vec{\nabla}_{\vec{r}} \times \vec{\sigma}_s(\vec{r})), \quad (95)$$

$$\vec{S}_s(\vec{r}) = \frac{\delta E}{\delta \vec{\sigma}_s(\vec{r})} = -b_4 (\vec{\nabla}_{\vec{r}} \times \vec{j}(\vec{r})) - b'_4 (\vec{\nabla}_{\vec{r}} \times \vec{j}_s(\vec{r})). \quad (96)$$

for time-odd ones. The last line in (92) is the Coulomb contribution. It includes only proton density ( $s = p$ ). Since static ground-state time-odd densities are zero, the values  $\vec{\sigma}(\vec{r})$  and  $\vec{j}(\vec{r})$  in (95) and (96) are reduced to the time-dependent density variations (6). We assume that the functional derivatives (92)-(96) and the involved densities are have time dependent but, for the sake of simplicity, do not depict this.

Using (92)-(94), we get the time-even part of the mean field Hamiltonian (8)

$$\begin{aligned}
\hat{h}_+^s(\vec{r}) & = U_s(\rho, z) - \vec{\nabla} B_s(\rho, z) \vec{\nabla} - i \vec{W}_s(\rho, z) \cdot \vec{\nabla} \times \hat{\vec{\sigma}} \\
& = U_s(\rho, z) + \overleftarrow{\nabla} B_s(\rho, z) \vec{\nabla} - i \vec{W}_s(\rho, z) \cdot \vec{\nabla} \times \hat{\vec{\sigma}}.
\end{aligned} \quad (97)$$

If to implement the static densities and currents, one gets the ground-state Hamiltonian, i.e.  $\hat{h}_+^s \rightarrow \hat{h}_0^s$ . The ground-state mean field of even-even axial nuclei has no any contributions from the time-odd variables.

The part of the mean field Hamiltonian (8) from the time-odd densities reads

$$\begin{aligned}\hat{h}_-^s(\vec{r}) &= -\frac{i}{2}\{\vec{A}_s(\rho, z), \vec{\nabla}\} + \frac{1}{2}\{\vec{S}_s(\rho, z), \hat{\vec{\sigma}}\} \\ &= \frac{i}{2}(\vec{\nabla} \cdot \vec{A}_s(\rho, z) - \vec{A}_s(\rho, z) \cdot \vec{\nabla}) + \vec{S}_s(\rho, z) \cdot \hat{\vec{\sigma}}.\end{aligned}\quad (98)$$

It is used only for derivation of the response Hamiltonian. Hence the involved current and spin densities are represented merely by their time-dependent variations.

## B. Time-even response

For time-even densities, the second functional derivatives read

$$\begin{aligned}\frac{\delta^2 E}{\delta \rho_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} &= \frac{\delta U_s(\vec{r})}{\delta \rho_{s_1}(\vec{r}_1)} \\ &= \{b_0 - b'_0 \delta_{ss'} + (-b_2 + b'_2 \delta_{ss_1}) \Delta_{\vec{r}_1} + b_3(\alpha + 2)(\alpha + 1) \rho^\alpha(\vec{r}) \\ &\quad - b'_3 \left[ \frac{\alpha(\alpha - 1)}{3} \rho^{\alpha-2}(\vec{r}) \sum_{s_2} \rho_{s_2}^2(\vec{r}) + \frac{2\alpha}{3} \rho^{\alpha-1}(\vec{r}) (\rho_s(\vec{r}) + \rho_{s_1}(\vec{r})) + \delta_{ss_1} \frac{2}{3} \rho^\alpha(\vec{r}) \right] \\ &\quad - \delta_{s,s_1} \delta_{s,p} \frac{1}{3} \left( \frac{2}{\pi} \right)^{1/3} [\rho_p(\vec{r})]^{-2/3} \} \delta(\vec{r} - \vec{r}_1) \\ &\quad + \delta_{s,s_1} \delta_{s,p} \frac{e^2}{|\vec{r} - \vec{r}_1|},\end{aligned}\quad (99)$$

$$\frac{\delta^2 E}{\delta \tau_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} = \frac{\delta U_s(\vec{r})}{\delta \tau_{s_1}(\vec{r}_1)} = [b_1 - b'_1 \delta_{ss'}] \delta(\vec{r} - \vec{r}_1), \quad (100)$$

$$\frac{\delta^2 E}{\delta \vec{\mathfrak{S}}_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} = \frac{\delta U_s(\vec{r})}{\delta \vec{\mathfrak{S}}_{s_1}(\vec{r}_1)} = [b_4 + b'_4 \delta_{ss_1}] \vec{\nabla}_{\vec{r}_1} \delta(\vec{r} - \vec{r}_1), \quad (101)$$

$$\frac{\delta^2 E}{\delta \rho_{s_1}(\vec{r}_1) \delta \vec{\mathfrak{S}}_s(\vec{r})} = \frac{\delta \vec{W}_s(\vec{r})}{\delta \rho_{s_1}(\vec{r}_1)} = -[b_4 + b'_4 \delta_{ss_1}] \vec{\nabla}_{\vec{r}_1} \delta(\vec{r} - \vec{r}_1). \quad (102)$$

The last two terms in (99) represent the exchange and direct Coulomb contributions. The pairing second functional derivative is not presented here. The pairing contribution to the response is considered in Sec. II G.

Following (15), the operator  $\hat{X}_{sk}^{s'}(\vec{r})$  reads

$$\begin{aligned}\hat{X}_{s_1 k_1}^s(\vec{r}) &= i \sum_{\alpha_1} \int d\vec{r}_1 \left[ \frac{\delta^2 \mathcal{E}}{\delta J_{s_1}^{\alpha_1}(\vec{r}_1) \delta J_s^\alpha(\vec{r})} \right]_{J=\bar{J}} < 0 | [\hat{P}_{s_1 k_1}, \hat{J}_{s_1}^{\alpha_1}] | 0 > (\vec{r}_1) \hat{J}_s^\alpha(\vec{r}) \\ &= U_{s_1 k_1}^s(\vec{r}) \hat{\rho}_s(\vec{r}) + B_{s_1 k_1}^s(\vec{r}) \hat{\tau}_s(\vec{r}) + \vec{W}_{s_1 k_1}^s(\vec{r}) \hat{\vec{\mathfrak{S}}}_s(\vec{r}) + \delta_{s,s_1} \hat{X}_{s k_1}^{s(\text{pair})}\end{aligned}\quad (103)$$

and its matrix element is

$$\begin{aligned} \langle ij | \hat{X}_{s_1 k_1}^s | \tilde{0} \rangle &= \int d\vec{r} \{ U_{s_1 k_1}^s(\vec{r}) \rho_{ij}^s(\vec{r}) + B_{s_1 k_1}^s(\vec{r}) \tau_{ij}^s(\vec{r}) + \vec{W}_{s_1 k_1}^s(\vec{r}) \vec{\mathfrak{S}}_{ij}^s(\vec{r}) \} \\ &+ \delta_{s, s_1} \langle ij | \hat{X}_{s k_1}^{s(pair)} | \tilde{0} \rangle \end{aligned} \quad (104)$$

where  $\rho_{ij}^s(\vec{r})$ ,  $\tau_{ij}^s(\vec{r})$ , and  $\vec{\mathfrak{S}}_{ij}^s(\vec{r})$  are transition densities (D3)-(D5). Expressions for operator  $\hat{X}_{s k_1}^{s(pair)}$  and matrix element  $\langle ij | \hat{X}_{s k_1}^{s(pair)} | \tilde{0} \rangle$  are done in (63) and the Appendix G 2, respectively. The functions  $U_{s_1 k_1}^s(\vec{r})$ ,  $B_{s_1 k_1}^s(\vec{r})$ , and  $\vec{W}_{s_1 k_1}^s(\vec{r})$  read

$$\begin{aligned} U_{s_1 k_1}^s(\vec{r}) &= \int d\vec{r}_1 \left[ \frac{\delta^2 E}{\delta \rho_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} \rho_{X, s_1 k_1}(\vec{r}_1) + \frac{\delta^2 E}{\delta \tau_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} \tau_{X, s_1 k_1}(\vec{r}_1) \right. \\ &\quad \left. + \frac{\delta^2 E}{\delta \vec{\mathfrak{S}}_{s_1}(\vec{r}_1) \delta \rho_s(\vec{r})} \vec{\mathfrak{S}}_{X, s_1 k_1}(\vec{r}_1) \right] \end{aligned} \quad (105)$$

$$= \bar{U}_{s_1 k_1}^s(\rho, z) \cos \mu \theta,$$

$$B_{s_1 k_1}^s(\vec{r}) = \int d\vec{r}_1 \frac{\delta^2 E}{\delta \rho_{s_1}(\vec{r}_1) \delta \tau_s(\vec{r})} \rho_{X, s_1 k_1}(\vec{r}_1) = \bar{B}_{s_1 k_1}^s(\rho, z) \cos \mu \theta, \quad (106)$$

$$\begin{aligned} \vec{W}_{s_1 k_1}^s(\vec{r}) &= \int d\vec{r}_1 \left[ \frac{\delta^2 E}{\delta \rho_{s_1}(\vec{r}_1) \delta \vec{\mathfrak{S}}_s(\vec{r})} \rho_{X, s_1 k_1}(\vec{r}_1) + \frac{\delta^2 E}{\delta \tau_{s_1}(\vec{r}_1) \delta \vec{\mathfrak{S}}_s(\vec{r})} \tau_{X, s_1 k_1}(\vec{r}_1) \right. \\ &\quad \left. + \frac{\delta^2 E}{\delta \vec{\mathfrak{S}}_{s_1}(\vec{r}_1) \delta \vec{\mathfrak{S}}_s(\vec{r})} \vec{\mathfrak{S}}_{X, s_1 k_1}(\vec{r}_1) \right] \\ &= \vec{e}_\rho \bar{W}_{s_1 k_1; \rho}^s(\rho, z) \cos \mu \theta + \vec{e}_z \bar{W}_{s_1 k_1; z}^s(\rho, z) \cos \mu \theta + \vec{e}_\theta \bar{W}_{s_1 k_1; \theta}^s(\rho, z) \sin \mu \theta \end{aligned} \quad (107)$$

where the values  $\bar{U}_{s_1 k_1}^s(\rho, z)$ ,  $\bar{B}_{s_1 k_1}^s(\rho, z)$ ,  $\bar{W}_{s_1 k_1; \rho}^s(\rho, z)$ ,  $\bar{W}_{s_1 k_1; z}^s(\rho, z)$ , and  $\bar{W}_{s_1 k_1; \theta}^s(\rho, z)$  are given in the Appendix G 2.

Contribution of the Coulomb integral to the matrix element (104) reads

$$\begin{aligned} \langle i | \hat{X}_{s_1 k_1}^s | j \rangle_{Coul} &= \delta_{s_1, p} \delta_{s, p} e^2 \int d\vec{r} \int d\vec{r}_1 \frac{\rho_{X_{k_1}, p}(\rho_1, z_1) \cos \mu \theta_1}{|\vec{r} - \vec{r}_1|} \rho_{ij}^s(\vec{r}) \\ &= \delta_{s_1, p} \delta_{s, p} e^2 \int d\vec{r} U_{k_1}^{Coul}(\rho, z) \cos \mu \theta \rho_{ij}^s(\vec{r}) \end{aligned} \quad (108)$$

where  $U_{k_1}^{Coul}(\rho, z)$  is determined in (F9). It should be included to  $\bar{U}_{s_1 k_1}^s(\rho, z)$ , see (G8).

In our case, the singularity problem for the Coulomb interaction cannot be treated by the familiar methods. These methods mainly deal with spherical systems [34] or axial systems in the static ground state [36] and, in any case, do not master the specific structure of the dynamical response in axial systems, given in (108). Hence we propose in Appendix F the new prescription.

### C. Time-odd response

For time-odd densities, the second functional derivatives read

$$\frac{\delta^2 E}{\delta \vec{j}_{s_1}(\vec{r}_1) \delta \vec{j}_s(\vec{r})} = \frac{\delta \vec{A}_s(\vec{r})}{\delta \vec{j}_{s_1}(\vec{r}_1)} = 2[-b_1 + b'_1 \delta_{s,s_1}] \delta(\vec{r}_1 - \vec{r}) , \quad (109)$$

$$\frac{\delta^2 E}{\delta \vec{\sigma}_{s_1}(\vec{r}_1) \delta \vec{j}_s(\vec{r})} = \frac{\delta \vec{A}_s(\vec{r})}{\delta \vec{\sigma}_{s_1}(\vec{r}_1)} = -[b_4 + b'_4 \delta_{s,s_1}] \vec{\nabla}_{\vec{r}_1} \times \delta(\vec{r}_1 - \vec{r}) , \quad (110)$$

$$\frac{\delta^2 E}{\delta \vec{j}_{s_1}(\vec{r}_1) \delta \vec{\sigma}_s(\vec{r})} = \frac{\delta \vec{S}_s(\vec{r})}{\delta \vec{j}_{s_1}(\vec{r}_1)} = -[b_4 + b'_4 \delta_{s,s_1}] \vec{\nabla}_{\vec{r}_1} \times \delta(\vec{r}_1 - \vec{r}) . \quad (111)$$

Following (16), the operator  $\hat{Y}_{s_1 k_1}^{s'}(\vec{r})$  has the form

$$\begin{aligned} \hat{Y}_{s_1 k_1}^s(\vec{r}) &= i \sum_{\alpha \alpha_1} \int d\vec{r}_1 \left[ \frac{\delta^2 \mathcal{E}}{\delta J_{s_1}^{\alpha_1}(\vec{r}_1) \delta J_s^\alpha(\vec{r})} \right]_{J=\bar{J} < 0} |[\hat{Q}_{s_1 k_1}, \hat{J}_{s_1}^{\alpha_1}]|0 > (\vec{r}_1) \hat{J}_s^\alpha(\vec{r}) \\ &= \vec{A}_{s_1 k_1}^s(\vec{r}) \hat{j}_s(\vec{r}) + \vec{S}_{s_1 k_1}^s(\vec{r}) \hat{\sigma}_s(\vec{r}) . \end{aligned} \quad (112)$$

and its matrix element is

$$< ij | \hat{Y}_{s_1 k_1}^s | \tilde{0} > = \int d\vec{r} [\bar{A}_{s_1 k_1; \rho}^s(\vec{r}) j_{ij; \rho}^s(\vec{r}) + \bar{A}_{s_1 k_1; z}^s(\vec{r}) j_{ij; z}^s(\vec{r}) + \bar{S}_{s_1 k_1; \theta}^s(\vec{r}) s_{ij; \theta}^s(\vec{r})] \quad (113)$$

where

$$\begin{aligned} \vec{A}_{s_1 k_1}^s(\vec{r}) &= \int d\vec{r}_1 \left[ \frac{\delta^2 E}{\delta \vec{j}_{s_1}(\vec{r}_1) \delta \vec{j}_s(\vec{r})} \vec{j}_{Y, s_1 k_1} + \frac{\delta^2 E}{\delta \vec{\sigma}_{s_1}(\vec{r}_1) \delta \vec{j}_s(\vec{r})} \vec{S}_{Y, s_1 k_1} \right] \\ &= 2[-b_1 + b'_1 \delta_{s,s_1}] \vec{j}_{Y, s_1 k_1} - [b_4 + b'_4 \delta_{s,s_1}] (\vec{\nabla} \times \vec{S}_{Y, s_1 k_1}) \\ &= \vec{e}_\rho \bar{A}_{s_1 k_1; \rho}^s(\rho, z) \cos \mu\theta + \vec{e}_z \bar{A}_{s_1 k_1; z}^s(\rho, z) \cos \mu\theta + \vec{e}_\theta \bar{A}_{s_1 k_1; \theta}^s(\rho, z) \sin \mu\theta , \end{aligned} \quad (114)$$

$$\begin{aligned} \vec{S}_{s_1 k_1}^s(\vec{r}) &= \int d\vec{r}_1 \frac{\delta^2 E}{\delta \vec{j}_{s_1}(\vec{r}_1) \delta \vec{\sigma}_s(\vec{r})} \vec{j}_{Y, s_1 k_1} \\ &= -[b_4 + b'_4 \delta_{s,s_1}] (\vec{\nabla} \times \vec{j}_{Y, s_1 k_1}) \\ &= \vec{e}_\rho \bar{S}_{s_1 k_1; \rho}^s(\rho, z) \sin \mu\theta + \vec{e}_z \bar{S}_{s_1 k_1; z}^s(\rho, z) \sin \mu\theta + \vec{e}_\theta \bar{S}_{s_1 k_1; \theta}^s(\rho, z) \cos \mu\theta , \end{aligned} \quad (115)$$

and

$$(\vec{\nabla} \times \vec{S}_{Y, sk}) = \vec{e}_\rho \left[ \frac{\mu}{\rho} s_{Y, sk; z}(\rho, z) - \partial_z s_{Y, sk; \theta}(\rho, z) \right] \cos \mu\theta \quad (116)$$

$$\begin{aligned} &+ \vec{e}_z \left[ \frac{1}{\rho} \partial_\rho (\rho s_{Y, sk; \theta}) - \frac{\mu}{\rho} s_{Y, sk; \rho} \right] \cos \mu\theta + \vec{e}_\theta [\partial_z s_{Y, sk; \rho} - \partial_\rho s_{Y, sk; z}] \sin \mu\theta , \\ (\vec{\nabla} \times \vec{j}_{Y, sk}) &= \vec{e}_\rho \left[ -\frac{\mu}{\rho} j_{Y, sk; z}(\rho, z) - \partial_z j_{Y, sk; \theta}(\rho, z) \right] \sin \mu\theta \\ &+ \vec{e}_z \left[ \frac{1}{\rho} \partial_\rho (\rho j_{Y, sk; \theta}) + \frac{\mu}{\rho} j_{Y, sk; \rho} \right] \sin \mu\theta + \vec{e}_\theta [\partial_z j_{Y, sk; \rho} - \partial_\rho j_{Y, sk; z}] \cos \mu\theta . \end{aligned} \quad (117)$$

Expressions for  $\bar{A}_{Y,sk;\rho}^{s'}(\rho, z)$ ,  $\bar{A}_{Y,sk;z}^{s'}(\rho, z)$ ,  $\bar{A}_{Y,sk;\theta}^{s'}(\rho, z)$ ,  $\bar{S}_{Y,sk;\rho}^{s'}(\rho, z)$ ,  $\bar{S}_{Y,sk;z}^{s'}(\rho, z)$ , and  $\bar{S}_{Y,sk;\theta}^{s'}(\rho, z)$  are given in the Appendix G 3.

#### IV. CHOICE OF INITIAL OPERATORS

As was discussed in Sec. III, all SRPA operations start from initial (generating) operators  $\hat{Q}_{sk}$ , see the sequence of the model steps in (82). The SRPA formalism itself does not provide these operators. At the same time, their proper choice is crucial to get good convergence of the separable expansion (2) with a minimal number of separable operators. The choice should be simple and universal in the sense that it can be applied equally well to all modes and excitation channels.

We propose the choice inspired by physical arguments. The main idea is that the generating operators should explore different spatial regions of the nucleus, the surface as well as the interior. The leading scaling generator should have the form of the applied external field in the long-wave approximation, which is most sensitive to the surface of the system. Since nuclear collective motion dominates in the surface region, already this generator should provide a good description. Next generators should be localized more in the interior to describe an interplay of surface and volume vibrations. For  $E\lambda$ -excitations in spherical nuclei, the best set of the generators was found to be [10]

$$\hat{Q}_k(\vec{r}) = R_k(r)(Y_{\lambda\mu}(\Omega) + Y_{\lambda\mu}^*(\Omega)) \quad , \quad \hat{P}_k = i[\hat{H}, \hat{Q}_k] \quad (118)$$

with

$$R_k(r) = \begin{cases} r^\lambda, & k=1 \\ j_\lambda(q_\lambda^k r), & k=2, 3, 4 \end{cases} \quad (119)$$

$$q_\lambda^k = a_k \frac{z_\lambda}{R_{\text{diff}}}, \quad a_2=0.6, a_3=0.9, a_4=1.2$$

where  $R_{\text{diff}}$  is the diffraction radius of the actual nucleus and  $z_\lambda$  is the first root in  $j_\lambda(z_\lambda) = 0$ . The separable operators  $\hat{X}_k$  and  $\hat{X}_k$  with  $k=1$  are mainly localized at the nuclear surface while the operators with  $k>1$  are localized more and more in the interior. This simple set seems to be a best compromise for the description of nuclear giant resonances in light and heavy nuclei.

In deformed nuclei, we exploit even simpler set with

$$R_k(r) = r^{\lambda+2(k-1)}. \quad (120)$$



Similar to the previous set, the separable operators  $\hat{X}_k$  and  $\hat{X}_k$  with  $k = 1$  are mainly localized at the nuclear surface while the next ones are localized more and more in the interior. We expect that already the first generators with  $k = 1$  and 2 are quite enough for description of giant resonances.

In the case of magnetic modes, the initial generators should be the time-odd. Following the same logic, the leading ( $k=1$ ) operator should coincide with the transition operator for the external magnetic field in the long-wave approximation. The next generators can be produced following the prescription (120) for the radial parts. The time-even conjugates of the time-odd generators can be obtained as  $\hat{Q}_k = i[\hat{H}, \hat{P}_k]_{ph}$ .

## V. CONCLUSIONS

A general procedure for self-consistent factorization of the residual nuclear interaction is proposed for arbitrary density- and current-dependent functionals. The separable RPA (SRPA) constructed in the framework of this approach can dramatically simplify the calculations while keeping high accuracy of numerical results. The economical effect of SRPA is especially actual for deformed nuclei. In the present contribution, SRPA is specified for description of axial nuclei with Skyrme forces.

SRPA can be used for description of  $E\lambda$  (and  $M\lambda$ ) response in both spherical and deformed nuclei. The approach can also serve for getting the basis of one-phonon RPA states for further description of anharmonic corrections, vibrational admixtures in the low-energy states in odd and odd-odd nuclei, etc. One of the most promising lines of future studies is dynamics of exotic nuclei obtained in radioactive beams.

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## APPENDIX A: CYLINDRICAL COORDINATES

### 1. Expressions for cylindrical coordinates

Cylindrical coordinates are defined as

$$\begin{aligned} x &= \rho \cos \vartheta & \vec{e}_x &= \vec{e}_\rho \cos \vartheta - \vec{e}_\vartheta \sin \vartheta & \vec{e}_\rho &= \vec{e}_x \cos \vartheta + \vec{e}_y \sin \vartheta \\ y &= \rho \sin \vartheta & \vec{e}_y &= \vec{e}_\rho \sin \vartheta + \vec{e}_\vartheta \cos \vartheta & \vec{e}_\vartheta &= -\vec{e}_x \sin \vartheta + \vec{e}_y \cos \vartheta \\ z &= z & \vec{e}_z &= \vec{e}_z & \vec{e}_z &= \vec{e}_z \end{aligned}$$

The gradient operator reads

$$\vec{\nabla} = \vec{e}_\rho \nabla_\rho + \vec{e}_\vartheta \nabla_\vartheta + \vec{e}_z \nabla_z$$

where

$$\nabla_\rho = \partial_\rho, \quad \nabla_\vartheta = \frac{1}{\rho} \partial_\vartheta = \frac{i \hat{L}_z}{\rho}, \quad \nabla_z = \partial_z$$

and  $\hat{L}_z = -i \partial_\vartheta$  is the third component of the orbital momentum operator.

The Laplacian, divergence, and curl read [37]

$$\begin{aligned} \Delta &= \vec{\nabla} \cdot \vec{\nabla} = \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial_\vartheta^2 + \partial_z^2, \\ \text{div} \vec{A} &= \vec{\nabla} \cdot \vec{A} = \frac{1}{\rho} \partial_\rho (\rho A_\rho) + \frac{1}{\rho} \partial_\vartheta A_\vartheta + \partial_z A_z, \\ \text{rot} \vec{A} &= \left[ \frac{1}{\rho} \partial_\vartheta A_z - \partial_z A_\vartheta \right] \vec{e}_\rho + \left[ \partial_z A_\rho - \partial_\rho A_z \right] \vec{e}_\vartheta \\ &\quad + \left[ \frac{1}{\rho} \partial_\rho (\rho A_\vartheta) - \frac{1}{\rho} \partial_\vartheta A_\rho \right] \vec{e}_z. \end{aligned}$$

The vector of Pauli matrices is

$$\hat{\vec{\sigma}} = \vec{e}_\rho \hat{\sigma}_\rho + \vec{e}_\vartheta \hat{\sigma}_\vartheta + \vec{e}_z \hat{\sigma}_z,$$

where

$$\hat{\sigma}_\rho = \begin{pmatrix} 0 & e^{-i\vartheta} \\ e^{i\vartheta} & 0 \end{pmatrix}, \quad \hat{\sigma}_\vartheta = i \begin{pmatrix} 0 & -e^{-i\vartheta} \\ e^{i\vartheta} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## 2. Single-particle wave function in cylindrical coordinates

The single-particle particle wave function and its time reversal are expressed in the cylindrical coordinates as spinors

$$\Psi_i(\vec{r}) = \begin{pmatrix} R_i^{(+)}(\rho, z)e^{im_i^{(+)}\vartheta} \\ R_i^{(-)}(\rho, z)e^{im_i^{(-)}\vartheta} \end{pmatrix}, \quad (\text{A1})$$

$$\Psi_{\bar{i}}(\vec{r}) = \hat{T}\Psi_i(\vec{r}) = \begin{pmatrix} -R_i^{(-)}(\rho, z)e^{-im_i^{(-)}\vartheta} \\ R_i^{(+)}(\rho, z)e^{-im_i^{(+)}\vartheta} \end{pmatrix} \quad (\text{A2})$$

where  $K_i$  is the projection of the complete single-particle moment onto symmetry z-axis of the axial nucleus.

In short notations covering both the state  $i$  and time-inverse  $\bar{i}$  states, the spinors read

$$\tilde{R}_i^{(\sigma)}(\rho, z)e^{im_i^{(\sigma)}\vartheta} = \begin{pmatrix} R_i^{(\sigma)}(\rho, z)e^{im_i^{(\sigma)}\vartheta} & \text{for ordinary state } i \\ -\sigma R_i^{(-\sigma)}(\rho, z)e^{-im_i^{(-\sigma)}\vartheta} & \text{for time reversal state } \bar{i} \end{pmatrix} \quad (\text{A3})$$

where

$$m_i^{(\sigma)} = K - \frac{1}{2}\sigma, \quad m_i^{(-\sigma)} = m_i^{(\sigma)} + \sigma, \quad m_i^{(\sigma)} = -m_i^{(-\sigma)}. \quad (\text{A4})$$

## APPENDIX B: USEFUL RELATIONS

### 1. Hermitian and time-conjugate properties

All the operators used in the model are hermitian ( $\hat{A} = \hat{A}^\dagger$ ) and have the definite time density

$$\hat{\tilde{A}} = T\hat{A}T^{-1} = \gamma_T^A \hat{A}, \quad \gamma_T^A = \pm 1$$

where  $T$  is the time-inversion operator:  $T|i\rangle = |\bar{i}\rangle$ ,  $T|\bar{i}\rangle = -|i\rangle$ .

There are useful transmutation relation for the hermitian operators

$$\langle j|\hat{A}^\dagger|i\rangle = \langle j|\hat{A}|i\rangle = \langle i|\hat{A}|j\rangle^* \quad (\text{B1})$$

and relations for time-inverse states and operators

$$\begin{aligned} \langle \bar{j}|\bar{A}|\bar{i}\rangle &= \langle \bar{j}|T^{-1}T\bar{A}T^{-1}T|\bar{i}\rangle = \langle T\bar{j}|T\bar{A}T^{-1}T|\bar{i}\rangle^* = \langle j|A|i\rangle^*, \\ \langle \bar{j}|\bar{A}|i\rangle &= -\langle j|A|\bar{i}\rangle^*. \end{aligned} \quad (\text{B2})$$

In axial even-even nuclei, the densities are built from the pairs

$$\langle i|A|i \rangle + \langle \bar{i}|A|\bar{i} \rangle = \langle i|A|i \rangle + \gamma_T^A \langle i|A|i \rangle^* = \langle i|A|i \rangle (1 + \gamma_T^A) \quad (\text{B3})$$

and hence vanish for time-odd operators.

## 2. Connections for operators with definite time-parity

Operators with the definite time-parity have the useful property

$$\langle 0|[\hat{A}, \hat{B}]|0 \rangle = \langle 0|\hat{A}\hat{B}|0 \rangle (1 - \gamma_T^A \gamma_T^B) \quad (\text{B4})$$

i.e. the average value of the commutator survives only if operators  $\hat{A}$  and  $\hat{B}$  are of the opposite time-parity. Indeed,

$$\begin{aligned} & \langle 0|[\hat{A}, \hat{B}]|0 \rangle = \langle 0|\hat{A}\hat{B}|0 \rangle - \langle 0|\hat{B}\hat{A}|0 \rangle \\ &= \langle 0|\hat{A}\hat{B}|0 \rangle - \langle 0|T^{-1}T\hat{B}T^{-1}T\hat{A}T^{-1}T|0 \rangle = \langle 0|\hat{A}\hat{B}|0 \rangle - \gamma_T^A \gamma_T^B \langle 0|T^{-1}\hat{B}\hat{A}T|0 \rangle \\ &= \langle 0|\hat{A}\hat{B}|0 \rangle - \gamma_T^A \gamma_T^B \langle T0|\hat{B}\hat{A}|T0 \rangle^* = \langle 0|\hat{A}\hat{B}|0 \rangle - \gamma_T^A \gamma_T^B \langle 0|\hat{B}\hat{A}|0 \rangle^* \\ &= \langle 0|\hat{A}\hat{B}|0 \rangle - \gamma_T^A \gamma_T^B \langle 0|(\hat{B}\hat{A})^+|0 \rangle = \langle 0|\hat{A}\hat{B}|0 \rangle (1 - \gamma_T^A \gamma_T^B). \end{aligned} \quad (\text{B5})$$

## 3. Representation through matrix elements

Responses (15)-(16) and inverse strength matrices (21)-(22) read as the averaged commutators

$$\langle 0|[\hat{A}, \hat{B}]|0 \rangle \quad \text{with} \quad \gamma_T^A = -\gamma_T^B. \quad (\text{B6})$$

Calculation of these values can be greatly simplified if to express them through the matrix elements of the operators  $\hat{A}$  and  $\hat{B}$ , because in practice these matrix elements are already in our disposal. In this case, the commutator reads

$$\begin{aligned} \langle 0|[\hat{A}_s, \hat{B}_s]|0 \rangle &= \sum_{ij} \{ \langle 0|\hat{A}_s|ij \rangle \langle ij|\hat{B}_s|0 \rangle - \langle 0|\hat{B}_s|ij \rangle \langle ij|\hat{A}_s|0 \rangle \} \\ &= 2 \sum_{ij}^{K_i, K_j > 0} \{ \langle ij|\hat{A}_s|0 \rangle^* \langle ij|\hat{B}_s|0 \rangle - \langle ij|\hat{A}_s|0 \rangle \langle ij|\hat{B}_s|0 \rangle^* \}. \end{aligned} \quad (\text{B7})$$

The sum runs all the states, both ordinary and time reversed. Using the properties (B1)-(B2), the matrix elements  $\langle \bar{i}j|\hat{A}_s|0 \rangle$  and  $\langle \bar{i}j|\hat{A}_s|0 \rangle$  are reduced to  $\langle ij|\hat{A}_s|0 \rangle$  and  $\langle i\bar{j}|\hat{A}_s|0 \rangle$ , respectively. Hence the coefficient 2.

In the SRPA, the operator  $\hat{A}$  is associated with operators  $\hat{Q}$  or  $\hat{P}$ , which have real or imagine matrix elements, respectively. Then

$$\begin{aligned} \langle 0 | [\hat{Q}_{sk}, \hat{B}_s] | 0 \rangle &= 2 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | 0 \rangle (\langle ij | \hat{B}_s | 0 \rangle - \langle ij | \hat{B}_s | 0 \rangle^*) \\ &= 4i \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | 0 \rangle \Im \{ \langle ij | \hat{B}_s | 0 \rangle \}, \end{aligned} \quad (\text{B8})$$

$$\begin{aligned} \langle 0 | [\hat{P}_{sk}, \hat{B}_s] | 0 \rangle &= -2 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{P}_{sk} | 0 \rangle (\langle ij | \hat{B}_s | 0 \rangle + \langle ij | \hat{B}_s | 0 \rangle^*) \\ &= -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{P}_{sk} | 0 \rangle \Re \{ \langle ij | \hat{B}_s | 0 \rangle \} \end{aligned} \quad (\text{B9})$$

where  $\Im\{\dots\}$  and  $\Re\{\dots\}$  mean the imagine and real parts of the values in the parenthesis.

Then the strength matrices read

$$\begin{aligned} \kappa_{s'k', sk}^{-1} &= -i \langle 0 | [\hat{P}_{s'k'}, \hat{X}_{sk}^{s'}] | 0 \rangle = 4i \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{P}_{s'k'} | 0 \rangle \Re \{ \langle ij | \hat{X}_{sk}^{s'} | 0 \rangle \} \\ &= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{s'k'} | 0 \rangle} \overline{\langle ij | \hat{X}_{sk}^{s'} | 0 \rangle}, \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} \eta_{s'k', sk}^{-1} &= -i \langle 0 | [\hat{Q}_{k'}, \hat{Y}_k] | 0 \rangle = 4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{s'k'} | 0 \rangle \Im \{ \langle ij | \hat{Y}_{sk}^{s'} | 0 \rangle \} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{s'k'} | 0 \rangle} \overline{\langle ij | \hat{Y}_{sk}^{s'} | 0 \rangle} \end{aligned} \quad (\text{B11})$$

where the overline matrix elements

$$\overline{\langle ij | \hat{P}_{s'k'} | 0 \rangle} = -i \langle ij | \hat{P}_{s'k'} | 0 \rangle, \quad \overline{\langle ij | \hat{X}_{sk}^{s'} | 0 \rangle} = \langle ij | \hat{X}_{sk}^{s'} | 0 \rangle, \quad (\text{B12})$$

$$\overline{\langle ij | \hat{Q}_{s'k'} | 0 \rangle} = \langle ij | \hat{Q}_{s'k'} | 0 \rangle, \quad \overline{\langle ij | \hat{Y}_{sk}^{s'} | 0 \rangle} = -i \langle ij | \hat{Y}_{sk}^{s'} | 0 \rangle \quad (\text{B13})$$

are real. It is seen that both strength matrices (B10) and (B11) are real as well.

The case of responses is more involved in the sense that matrix elements of the second operator in the commutator are transition densities (D1)-(D5) which are generally complex. The responses entering  $\hat{X}$  and  $\hat{Y}$  operators read

$$\mathcal{R}_{X, sk}^\alpha = i \langle 0 | [\hat{P}_{sk}, \hat{J}_s^\alpha] | 0 \rangle = -4i \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{P}_{sk} | 0 \rangle \Re \{ \langle ij | \hat{J}_s^\alpha | 0 \rangle \}$$

$$= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | 0 \rangle} \Re \{ \langle ij | \hat{J}_s^\alpha | 0 \rangle \}, \quad (\text{B14})$$

$$\begin{aligned} \mathcal{R}_{Y,sk}^\alpha &= i \langle 0 | [\hat{Q}_{sk}, \hat{J}_s^\alpha] | 0 \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | 0 \rangle \Im \{ \langle ij | \hat{J}_s^\alpha | 0 \rangle \} \\ &= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | 0 \rangle} \Im \{ \langle ij | \hat{J}_s^\alpha | 0 \rangle \} \end{aligned} \quad (\text{B15})$$

where  $\langle ij | \hat{J}_s^\alpha | 0 \rangle$  are transition densities. It is seen that all the responses are real.

Following (D1)-(D5), the transition densities (or their components) for  $\mu = K_i \pm K_j$  are proportional to  $e^{-i\mu\vartheta} = \cos \mu\vartheta - i \sin \mu\vartheta$  or  $ie^{-i\mu\vartheta} = i \cos \mu\vartheta + \sin \mu\vartheta$ . Hence the response obviously vanishes at  $\mu = 0$ , if its  $\Im\{\dots\}$  or  $\Re\{\dots\}$  part delivers  $\sin \mu\vartheta$ .

### APPENDIX C: DENSITIES AND CURRENTS FOR SKYRME FUNCTIONAL

In Skyrme forces, the complete set of the densities involves the ordinary density, kinetic-energy density, spin-orbital density, current density and spin density:

$$\begin{aligned} \rho_s(\vec{r}, t) &= \sum_{h \in s}^{occ} \varphi_h^*(\vec{r}, t) \varphi_h(\vec{r}, t), & \hat{T} \rho \hat{T}^{-1} &= \rho \\ \tau_s(\vec{r}, t) &= \sum_{h \in s}^{occ} \vec{\nabla} \varphi_h^*(\vec{r}, t) \cdot \vec{\nabla} \varphi_h(\vec{r}, t), & \hat{T} \tau \hat{T}^{-1} &= \tau \\ \vec{\mathfrak{S}}_s(\vec{r}, t) &= -i \sum_{h \in s}^{occ} \varphi_h^*(\vec{r}, t) (\vec{\nabla} \times \hat{\sigma}) \varphi_h(\vec{r}, t), & \hat{T} \vec{\mathfrak{S}} \hat{T}^{-1} &= \vec{\mathfrak{S}} \\ \vec{j}_s(\vec{r}, t) &= \frac{-i}{2} \sum_{h \in s}^{occ} \left[ \varphi_h^*(\vec{r}, t) \vec{\nabla} \varphi_h(\vec{r}, t) - \vec{\nabla} \varphi_h^*(\vec{r}, t) \varphi_h(\vec{r}, t) \right], & \hat{T} \vec{j} \hat{T}^{-1} &= -\vec{j} \\ \vec{\sigma}_s(\vec{r}) &= \sum_{h \in s}^{occ} \varphi_h^*(\vec{r}, t) \hat{\sigma} \varphi_h(\vec{r}, t), & \hat{T} \vec{\sigma} \hat{T}^{-1} &= -\vec{\sigma} \end{aligned}$$

where the sum runs over the occupied (hole) single-particle states  $h$ . The associated *hermitian* operators are

$$\begin{aligned} \hat{\rho}_s(\vec{r}) &= \sum_{i=1}^{N_s} \delta(\vec{r}_i - \vec{r}), \\ \hat{\tau}_s(\vec{r}) &= \sum_{i=1}^{N_s} \overleftarrow{\nabla} \delta(\vec{r}_i - \vec{r}) \vec{\nabla}, \end{aligned}$$

$$\begin{aligned}
\hat{\mathfrak{S}}_s(\vec{r}) &= \sum_{i=1}^{N_s} \delta(\vec{r}_i - \vec{r}) \vec{\nabla} \times \hat{\vec{\sigma}}, \\
\hat{\vec{j}}_s(\vec{r}) &= \frac{1}{2} \sum_{i=1}^{N_s} \left\{ \vec{\nabla}, \delta(\vec{r}_i - \vec{r}) \right\}, \\
\hat{\vec{\sigma}}_s(\vec{r}) &= \sum_{i=1}^{N_s} \delta(\vec{r}_i - \vec{r}) \hat{\vec{\sigma}}.
\end{aligned}$$

where  $\hat{\vec{\sigma}}$  is the Pauli matrix,  $N_s$  is number of protons or neutrons in the nucleus.

## APPENDIX D: TRANSITION DENSITIES

### 1. Explicit expressions

For  $\mu = |K_i - K_j|$  with  $K_i, K_j > 0$ , the transition densities read

$$\langle ij | \hat{j}_s | \tilde{0} \rangle(\vec{r}) = [\vec{e}_\rho \cdot ij_{ij; \rho}^s(\rho, z) + \vec{e}_z \cdot ij_{ij; z}^s(\rho, z) + \vec{e}_\vartheta \cdot ij_{ij; \vartheta}^s(\rho, z)] e^{-i(K_i - K_j)\vartheta}, \quad (\text{D1})$$

$$\langle ij | \hat{s}_s | \tilde{0} \rangle(\vec{r}) = [\vec{e}_\rho \cdot s_{ij; \rho}^s(\rho, z) + \vec{e}_z \cdot s_{ij; z}^s(\rho, z) + \vec{e}_\vartheta \cdot is_{ij; \vartheta}^s(\rho, z)] e^{-i(K_i - K_j)\vartheta}, \quad (\text{D2})$$

$$\langle ij | \hat{\rho}_s | \tilde{0} \rangle(\vec{r}) = \rho_{ij}^s(\rho, z) e^{-i(K_i - K_j)\vartheta}, \quad (\text{D3})$$

$$\langle ij | \hat{\tau}_s | \tilde{0} \rangle(\vec{r}) = \tau_{ij}^s(\rho, z) e^{-i(K_i - K_j)\vartheta}, \quad (\text{D4})$$

$$\langle ij | \hat{\mathfrak{S}}_s | \tilde{0} \rangle(\vec{r}) = [\vec{e}_\rho \cdot \mathfrak{S}_{ij; \rho}^s(\rho, z) + \vec{e}_z \cdot \mathfrak{S}_{ij; z}^s(\rho, z) + \vec{e}_\vartheta \cdot i\mathfrak{S}_{ij; \vartheta}^s(\rho, z)] e^{-i(K_i - K_j)\vartheta} \quad (\text{D5})$$

where the components with low indices  $ij$  are real. The case of the time-inverse state  $\bar{j}$  is straightforwardly obtained by  $K_j \rightarrow -K_j$ . As is shown below, the components accompanied by the imagine unit ( $j_{ij; \rho}^s, j_{ij; z}^s, s_{ij; \vartheta}^s$ , and  $\mathfrak{S}_{ij; \vartheta}^s$ ) demonstrate the specific features.

Using wave functions from the Appendix A 2 and density operators from the Appendix C, one gets expressions for the real components of the transition densities (D1)-(D5) and for their transmutation properties:

$$j_{ij; \rho}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \left[ (\partial_\rho \tilde{R}_i^{(\sigma)}) \tilde{R}_j^{(\sigma)} - \tilde{R}_i^{(\sigma)} (\partial_\rho \tilde{R}_j^{(\sigma)}) \right], \quad (\text{D6})$$

$$j_{ij; z}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \left[ (\partial_z \tilde{R}_i^{(\sigma)}) \tilde{R}_j^{(\sigma)} - \tilde{R}_i^{(\sigma)} (\partial_z \tilde{R}_j^{(\sigma)}) \right], \quad (\text{D7})$$

$$j_{ij; \vartheta}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \left[ \frac{\tilde{R}_i^{(\sigma)} \tilde{R}_j^{(\sigma)}}{\rho} (\tilde{m}_i^{(\sigma)} + \tilde{m}_j^{(\sigma)}) \right]. \quad (\text{D8})$$

$$j_{ji; \rho}^s(\rho, z) = j_{ij; \rho}^s(\rho, z), \quad j_{ji; z}^s(\rho, z) = j_{ij; z}^s(\rho, z), \quad j_{ji; \vartheta}^s(\rho, z) = -j_{ij; \vartheta}^s(\rho, z).$$

$$s_{ij;\rho}^s(\rho, z) = \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \left[ \tilde{R}_i^{(-\sigma)} \tilde{R}_j^{(\sigma)} \right], \quad (\text{D9})$$

$$s_{ij;z}^s(\rho, z) = \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \sigma \left[ \tilde{R}_i^{(\sigma)} \tilde{R}_j^{(\sigma)} \right], \quad (\text{D10})$$

$$s_{ij;\vartheta}^s(\rho, z) = \mathcal{U}_{ij}^{(-)} \sum_{\sigma=+,-} \sigma \left[ \tilde{R}_i^{(-\sigma)} \tilde{R}_j^{(\sigma)} \right]. \quad (\text{D11})$$

$$s_{ji;\rho}^s(\rho, z) = -s_{ij;\rho}^s(\rho, z), \quad s_{ji;z}^s(\rho, z) = -s_{ij;z}^s(\rho, z), \quad s_{ji;\vartheta}^s(\rho, z) = s_{ij;\vartheta}^s(\rho, z).$$

$$\rho_{ij}^s(\rho, z) = \mathcal{U}_{ij}^{(+)} \sum_{\sigma=+,-} \left[ \tilde{R}_i^{(\sigma)} \tilde{R}_j^{(\sigma)} \right]. \quad (\text{D12})$$

$$\rho_{ji}^s(\rho, z) = \rho_{ij}^s(\rho, z).$$

$$\tau_{ij}^s(\rho, z) = \mathcal{U}_{ij}^{(+)} \sum_{\sigma=+,-} \left[ (\partial_\rho \tilde{R}_i^{(\sigma)}) (\partial_\rho \tilde{R}_j^{(\sigma)}) + (\partial_z \tilde{R}_i^{(\sigma)}) (\partial_z \tilde{R}_j^{(\sigma)}) + \frac{\tilde{m}_i^{(\sigma)} \tilde{m}_j^{(\sigma)}}{\rho^2} \tilde{R}_i^{(\sigma)} \tilde{R}_j^{(\sigma)} \right]. \quad (\text{D13})$$

$$\tau_{ji}^s(\rho, z) = \tau_{ij}^s(\rho, z).$$

$$\begin{aligned} \mathfrak{S}_{ij;\rho}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(+)} \sum_{\sigma=+,-} \sigma \left[ \tilde{R}_i^{(\sigma)} (\partial_z \tilde{R}_j^{(-\sigma)}) + \tilde{R}_j^{(\sigma)} (\partial_z \tilde{R}_i^{(-\sigma)}) \right. \\ \left. + (\tilde{m}_i^{(\sigma)} + \tilde{m}_j^{(\sigma)}) \frac{\tilde{R}_i^{(\sigma)} \tilde{R}_j^{(\sigma)}}{\rho} \right], \end{aligned} \quad (\text{D14})$$

$$\begin{aligned} \mathfrak{S}_{ij;z}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(+)} \sum_{\sigma=+,-} \left[ \sigma \left( \tilde{R}_i^{(-\sigma)} (\partial_\rho \tilde{R}_j^{(\sigma)}) + \tilde{R}_j^{(-\sigma)} (\partial_\rho \tilde{R}_i^{(\sigma)}) \right) \right. \\ \left. - (\tilde{m}_i^{(-\sigma)} + \tilde{m}_j^{(\sigma)}) \frac{\tilde{R}_i^{(\sigma)} \tilde{R}_j^{(-\sigma)}}{\rho} \right], \end{aligned} \quad (\text{D15})$$

$$\begin{aligned} \mathfrak{S}_{ij;\vartheta}^s(\rho, z) = \frac{1}{2} \mathcal{U}_{ij}^{(+)} \sum_{\sigma=+,-} \left[ \sigma \left( \tilde{R}_i^{(\sigma)} (\partial_\rho \tilde{R}_j^{(\sigma)}) + \tilde{R}_j^{(-\sigma)} (\partial_\rho \tilde{R}_i^{(-\sigma)}) \right) \right. \\ \left. - \tilde{R}_i^{(\sigma)} (\partial_z \tilde{R}_j^{(-\sigma)}) + \tilde{R}_j^{(-\sigma)} (\partial_z \tilde{R}_i^{(\sigma)}) \right]. \end{aligned} \quad (\text{D16})$$

$$\mathfrak{S}_{ji;\rho}^s(\rho, z) = \mathfrak{S}_{ij;\rho}^s(\rho, z), \quad \mathfrak{S}_{ji;z}^s(\rho, z) = \mathfrak{S}_{ij;z}^s(\rho, z), \quad \mathfrak{S}_{ji;\vartheta}^s(\rho, z) = -\mathfrak{S}_{ij;\vartheta}^s(\rho, z).$$

## 2. Useful features

Some components of the vector responses become zero at  $\mu = 0$ . These cases can be revealed by inspecting the specific combinations of the transition densities involved into



responses. Following (B14)-(B15), these combinations are

$$\langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle + \langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle^* = 2\Re\{\langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle\}$$

for time-even densities

$$\langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle - \langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle^* = 2i\Im\{\langle ij|\hat{J}_s^\alpha|\tilde{0}\rangle\}$$

and for time-odd ones.

Specifically, these combinations read

$$\begin{aligned} & \langle ij|\hat{\vec{J}}_s|\tilde{0}\rangle - \langle ij|\hat{\vec{J}}_s|\tilde{0}\rangle^* \\ &= 2i[(\vec{e}_\rho \cdot \vec{j}_{ij;\rho}^s(\rho, z) + \vec{e}_z \cdot \vec{j}_{ij;z}^s(\rho, z)) \cos \mu\vartheta - \vec{e}_\vartheta \cdot \vec{j}_{ij;\vartheta}^s(\rho, z) \sin \mu\vartheta], \\ & \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle - \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle^* \\ &= 2i[-(\vec{e}_\rho \cdot \vec{s}_{ij;\rho}^s(\rho, z) + \vec{e}_z \cdot \vec{s}_{ij;z}^s(\rho, z)) \sin \mu\vartheta + \vec{e}_\vartheta \cdot \vec{s}_{ij;\vartheta}^s(\rho, z) \cos \mu\vartheta]. \\ & \langle ij|\hat{\rho}_s|\tilde{0}\rangle + \langle ij|\hat{\rho}_s|\tilde{0}\rangle^* = 2\rho_{ij}^s(\rho, z) \cos \mu\vartheta, \\ & \langle ij|\hat{\tau}_s|\tilde{0}\rangle + \langle ij|\hat{\tau}_s|\tilde{0}\rangle^* = 2\tau_{ij}^s(\rho, z) \cos \mu\vartheta, \\ & \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle + \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle^* \\ &= 2[(\vec{e}_\rho \cdot \vec{\mathfrak{S}}_{ij;\rho}^s(\rho, z) + \vec{e}_z \cdot \vec{\mathfrak{S}}_{ij;z}^s(\rho, z)) \cos \mu\vartheta - \vec{e}_\vartheta \cdot \vec{\mathfrak{S}}_{ij;\vartheta}^s(\rho, z) \sin \mu\vartheta]. \end{aligned}$$

For  $\mu = 0$ , the combinations with  $\sin \mu\vartheta$  vanish

$$(\langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle + \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle^*)_\theta = 0, \quad (\langle ij|\hat{\vec{J}}_s|\tilde{0}\rangle - \langle ij|\hat{\vec{J}}_s|\tilde{0}\rangle^*)_\theta = 0, \quad (\text{D17})$$

$$(\langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle - \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle^*)_\rho = 0, \quad (\langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle - \langle ij|\hat{\vec{S}}_s|\tilde{0}\rangle^*)_z = 0 \quad (\text{D18})$$

and hence the corresponding response components

$$\mu = 0 \quad \rightarrow \quad j_{Y,sk}^\theta = s_{Y,sk}^z = s_{Y,sk}^\rho = \mathfrak{S}_{Y,sk}^\theta = 0. \quad (\text{D19})$$

The curl in cylindrical coordinates is

$$\text{rot}\vec{A} = \nabla \times \vec{A} = \left[\frac{1}{\rho}\partial_\vartheta A_z - \partial_z A_\vartheta\right]\vec{e}_\rho + [\partial_z A_\rho - \partial_\rho A_z]\vec{e}_\vartheta + \left[\frac{1}{\rho}\partial_\rho(\rho A_\vartheta) - \frac{1}{\rho}\partial_\vartheta A_\rho\right]\vec{e}_z. \quad (\text{D20})$$

Then, taking into account (D17)-(D18) and zero value of the derivatives  $\partial_\theta$  from all the transition densities at  $\mu = 0$ , one gets

$$\begin{aligned}
\mu = 0 \quad \rightarrow \quad & \{\nabla \times (\langle ij|\vec{j}_s|\tilde{0} \rangle - \langle ij|\vec{j}_s|\tilde{0} \rangle^*)\}_\rho = 0, \\
& \{\nabla \times (\langle ij|\vec{j}_s|\tilde{0} \rangle - \langle ij|\vec{j}_s|\tilde{0} \rangle^*)\}_z = 0, \\
& \{\nabla \times (\langle ij|\vec{s}_s|\tilde{0} \rangle - \langle ij|\vec{s}_s|\tilde{0} \rangle^*)\}_\theta = 0, \\
& \{\nabla \times (\langle ij|\vec{\mathfrak{S}}_s|\tilde{0} \rangle + \langle ij|\vec{\mathfrak{S}}_s|\tilde{0} \rangle^*)\}_\rho = 0, \\
& \{\nabla \times (\langle ij|\vec{\mathfrak{S}}_s|\tilde{0} \rangle + \langle ij|\vec{\mathfrak{S}}_s|\tilde{0} \rangle^*)\}_z = 0.
\end{aligned} \tag{D21}$$

These relations are used in derivation of the first and second functional derivatives of the Skyrme functional (terms with  $b_4$  and  $b'_4$ ).

## APPENDIX E: RESPONSES

The responses have the general form (B14)-(B15). The explicit expressions read

$$\begin{aligned}
\vec{j}_{Y,sk}(\vec{r}) &= \vec{e}_\rho j_{Y,sk}^\rho(\vec{r}) + \vec{e}_z j_{Y,sk}^z(\vec{r}) + \vec{e}_\theta j_{Y,sk}^\theta(\vec{r}) \\
&= \vec{e}_\rho j_{Y,sk}^\rho(\rho, z) \cos \mu\theta + \vec{e}_z j_{Y,sk}^z(\rho, z) \cos \mu\theta + \vec{e}_\theta j_{Y,sk}^\theta(\rho, z) \sin \mu\theta, \\
\vec{s}_{Y,sk}(\vec{r}) &= \vec{e}_\rho s_{Y,sk}^\rho(\vec{r}) + \vec{e}_z s_{Y,sk}^z(\vec{r}) + \vec{e}_\theta s_{Y,sk}^\theta(\vec{r}) \\
&= \vec{e}_\rho s_{Y,sk}^\rho(\rho, z) \sin \mu\theta + \vec{e}_z s_{Y,sk}^z(\rho, z) \sin \mu\theta + \vec{e}_\theta s_{Y,sk}^\theta(\rho, z) \cos \mu\theta, \\
\rho_{X,sk}(\vec{r}) &= \rho_{X,sk}(\rho, z) \cos \mu\theta \\
\tau_{X,sk}(\vec{r}) &= \tau_{X,sk}(\rho, z) \cos \mu\theta \\
\vec{\mathfrak{S}}_{X,sk}(\vec{r}) &= \vec{e}_\rho \mathfrak{S}_{X,sk}^\rho(\vec{r}) + \vec{e}_z \mathfrak{S}_{X,sk}^z(\vec{r}) + \vec{e}_\theta \mathfrak{S}_{X,sk}^\theta(\vec{r}) \\
&= \vec{e}_\rho \mathfrak{S}_{X,sk}^\rho(\rho, z) \cos \mu\theta + \vec{e}_z \mathfrak{S}_{X,sk}^z(\rho, z) \cos \mu\theta + \vec{e}_\theta \mathfrak{S}_{X,sk}^\theta(\rho, z) \sin \mu\theta,
\end{aligned} \tag{E1}$$

The components of time-odd responses have the form

$$\begin{aligned}
j_{Y,sk}^\rho(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{j}_s^\rho] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle \mathfrak{S} \{ \langle ij | \hat{j}_s^\rho | \tilde{0} \rangle \} \\
&= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} j_{ij, \rho}^s(\rho, z) \cos \mu\theta = j_{Y,sk}^\rho(\rho, z) \cos \mu\theta, \\
j_{Y,sk}^z(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{j}_s^z] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle \mathfrak{S} \{ \langle ij | \hat{j}_s^z | \tilde{0} \rangle \}
\end{aligned} \tag{E3}$$

$$= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} j_{ij; z}^s(\rho, z) \cos \mu\theta = j_{Y,sk}^z(\rho, z) \cos \mu\theta, \quad (\text{E4})$$

$$\begin{aligned} j_{Y,sk}^\theta(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{j}_s^\theta] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle \Im\{\langle ij | \hat{j}_s^\theta | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} j_{ij; \theta}^s(\rho, z) \sin \mu\theta = j_{Y,sk}^\theta(\rho, z) \sin \mu\theta, \end{aligned} \quad (\text{E5})$$

$$\begin{aligned} s_{Y,sk}^\rho(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{s}_s^\rho] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle \Im\{\langle ij | \hat{s}_s^\rho | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} s_{ij; \rho}^s(\rho, z) \sin \mu\theta = s_{Y,sk}^\rho(\rho, z) \sin \mu\theta, \end{aligned} \quad (\text{E6})$$

$$\begin{aligned} s_{Y,sk}^z(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{s}_s^z] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle \Im\{\langle ij | \hat{s}_s^z | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} s_{ij; z}^s(\rho, z) \sin \mu\theta = s_{Y,sk}^z(\rho, z) \sin \mu\theta, \end{aligned} \quad (\text{E7})$$

$$\begin{aligned} s_{Y,sk}^\theta(\vec{r}) &= i \langle \tilde{0} | [\hat{Q}_{sk}, \hat{s}_s^\theta] | \tilde{0} \rangle = 2i \sum_{ij}^{K_i, K_j > 0} \langle ij | \hat{Q}_{sk} | \tilde{0} \rangle 2i \Im\{\langle ij | \hat{s}_s^\theta | \tilde{0} \rangle\} \\ &= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{Q}_{sk} | \tilde{0} \rangle} s_{ij; \theta}^s(\rho, z) \cos \mu\theta = s_{Y,sk}^\theta(\rho, z) \cos \mu\theta. \end{aligned} \quad (\text{E8})$$

The time-even responses and their components are

$$\begin{aligned} \rho_{X,sk}(\vec{r}) &= i \langle \tilde{0} | [\hat{P}_{sk}, \hat{\rho}_s] | \tilde{0} \rangle = -4i \sum_{ij}^{K_i, K_j > 0} \langle ij | [\hat{P}_{sk} | \tilde{0} \rangle \Re\{\langle ij | \hat{\rho}_s | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | \tilde{0} \rangle} \rho_{ij}^s(\rho, z) \cos \mu\theta = \rho_{X,sk}(\rho, z) \cos \mu\theta, \end{aligned} \quad (\text{E9})$$

$$\begin{aligned} \tau_{X,sk}(\vec{r}) &= i \langle \tilde{0} | [\hat{P}_{sk}, \hat{\tau}_s] | \tilde{0} \rangle = -4 \sum_{ij}^{K_i, K_j > 0} \langle ij | [\hat{P}_{sk} | \tilde{0} \rangle \Re\{\langle ij | \hat{\tau}_s | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | \tilde{0} \rangle} \tau_{ij}^s(\rho, z) \cos \mu\theta = \tau_{X,sk}(\rho, z) \cos \mu\theta, \end{aligned} \quad (\text{E10})$$

$$\begin{aligned} \Im_{Y,sk}^\rho(\vec{r}) &= i \langle \tilde{0} | [\hat{P}_{sk}, \hat{\Im}_s^\rho] | \tilde{0} \rangle = -4i \sum_{ij}^{K_i, K_j > 0} \langle ij | [\hat{P}_{sk} | \tilde{0} \rangle \Re\{\langle ij | \hat{\Im}_s^\rho | \tilde{0} \rangle\} \\ &= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | \tilde{0} \rangle} \Im_{ij; \rho}^s(\rho, z) \cos \mu\theta = \Im_{X,sk}^\rho(\rho, z) \cos \mu\theta, \end{aligned} \quad (\text{E11})$$

$$\begin{aligned}
\mathfrak{S}_{Y,sk}^z(\vec{r}) &= i \langle \tilde{0} | [\hat{P}_{sk}, \hat{\mathfrak{S}}_s^z] | \tilde{0} \rangle = -4i \sum_{ij}^{K_i, K_j > 0} \langle ij | [\hat{P}_{sk} | \tilde{0} \rangle \Re \{ \langle ij | \hat{\mathfrak{S}}_s^z | \tilde{0} \rangle \} \\
&= 4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | \tilde{0} \rangle} \mathfrak{S}_{ij; z}^s(\rho, z) \cos \mu\theta = \mathfrak{S}_{X,sk}^z(\rho, z) \cos \mu\theta, \quad (E12)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{S}_{Y,sk}^\theta(\vec{r}) &= i \langle \tilde{0} | [\hat{P}_{sk}, \hat{\mathfrak{S}}_s^\theta] | \tilde{0} \rangle = -4i \sum_{ij}^{K_i, K_j > 0} \langle ij | [\hat{P}_{sk} | \tilde{0} \rangle \Re \{ \langle ij | \hat{\mathfrak{S}}_s^\theta | \tilde{0} \rangle \} \\
&= -4 \sum_{ij}^{K_i, K_j > 0} \overline{\langle ij | \hat{P}_{sk} | \tilde{0} \rangle} \mathfrak{S}_{ij; \theta}^s(\rho, z) \sin \mu\theta = \mathfrak{S}_{X,sk}^\theta(\rho, z) \sin \mu\theta. \quad (E13)
\end{aligned}$$

The overline matrix elements are defined in (B12)-(B13). The pairing factors (60) are included into the matrix elements and transition densities. The values (E1)-(E13) are real. Time-even responses have diagonal contributions  $i = j$  while time-odd ones not.

For  $\mu = 0$ , the responses fulfill (D19) and

$$\begin{aligned}
(\nabla \times \vec{j}_{Y,sk})_\rho &= 0, \quad (\nabla \times \vec{j}_{Y,sk})_z = 0, \quad (\nabla \times \vec{j}_{Y,sk})_\vartheta = \partial_z j_{Y,sk}^\rho - \partial_\rho j_{Y,sk}^z, \\
(\nabla \times \vec{s}_{Y,sk})_\rho &= -\partial_z s_{Y,sk}^\vartheta, \quad (\nabla \times \vec{s}_{Y,sk})_z = \frac{1}{\rho} \partial_\rho (\rho s_{Y,sk}^\vartheta), \quad (\nabla \times \vec{s}_{Y,sk})_\vartheta = 0, \quad (E14) \\
(\nabla \times \vec{\mathfrak{S}}_{X,sk})_\rho &= 0, \quad (\nabla \times \vec{\mathfrak{S}}_{X,sk})_z = 0, \quad (\nabla \times \vec{\mathfrak{S}}_{X,sk})_\vartheta = \partial_z \mathfrak{S}_{X,sk}^\rho - \partial_\rho \mathfrak{S}_{X,sk}^z.
\end{aligned}$$

It is easy to see that the properties (D19) and (E14) of the responses fully repeat the properties (D17)-(D18) and (D21) of the combinations of the transition densities, entering the responses.

## APPENDIX F: COULOMB CONTRIBUTION

The contribution of the Coulomb integral to the SRPA response is

$$U_k^{Coul}(\vec{r}) = e^2 \int d\vec{r}' \frac{\rho_{X,pk}(\rho', z') \cos \mu\theta'}{|\vec{r} - \vec{r}'|}. \quad (F1)$$

The typical trouble is connected with the singularity at the point  $\vec{r} = \vec{r}'$ . Because of the angular dependence  $\cos \mu\theta'$  in the nominator of (F1), this trouble cannot be circumvented by common methods. For example, the FTT solver method [34] and procedures [35, 36] do not assume any angular dependence of this kind. So, we should develop our own prescription.

First, we bypass the logarithmic singularity by using the Vauterin identity [35, 36]

$$\Delta_{\vec{r}'}|\vec{r}-\vec{r}'| = \frac{2}{|\vec{r}-\vec{r}'|}. \quad (\text{F2})$$

Then, after integration by parts, the integral (F1) is reduced to

$$\begin{aligned} U_k^{Coul}(\vec{r}) &= \frac{e^2}{2} \int d\vec{r}' |\vec{r}-\vec{r}'| \Delta_{\vec{r}'} \rho_{X,pk}(\rho', z') \cos \mu\theta' \\ &= \frac{e^2}{2} \int \rho' d\rho' dz' [\partial_{\rho'}^2 + \frac{1}{\rho'} \partial_{\rho'} - \frac{\mu^2}{(\rho')^2} + \partial_{z'}^2] \rho_{X,pk}(\rho', z') \\ &\quad \cdot \int_0^{2\pi} d\theta' |\vec{r}-\vec{r}'| \cos \mu\theta'. \end{aligned} \quad (\text{F3})$$

For  $\mu = 0$ , the angular integral in (F3) is recast to

$$I(\vec{r}; \rho', z') = \int_0^{2\pi} d\theta' |\vec{r}-\vec{r}'| \cos \mu\theta' = 4\sqrt{d(\rho, z)} E\left(\frac{4\rho\rho'}{d(\rho, z)}\right) \quad (\text{F4})$$

where  $d(\rho, z) = (\rho + \rho')^2 + (z - z')^2$  and  $E(\frac{4\rho\rho'}{d(\rho, z)})$  is the elliptic integral of the second order. This integral can be approximated by a standard polynomial formula. However, in the general case  $\mu \geq 0$  we cannot get this result. So, we should develop another procedure.

First of all, we should take into account that all the terms in (105) have the common angular dependence  $\cos \mu\theta$ . This should be the case for the Coulomb term as well. To prove this, it is convenient to rewrite the angular integral from (F3) in the form

$$I(\vec{r}; \rho', z') = \int_0^{2\pi} d\theta' |\vec{r}-\vec{r}'| \cos \mu\theta' = I_1 \cos \mu\theta - I_2 \sin \mu\theta \quad (\text{F5})$$

where

$$I_1(\rho, z; \rho', z') = \int_0^{2\pi} d\theta' |\vec{r}-\vec{r}'| \cos \mu(\theta' - \theta), \quad I_2(\rho, z; \rho', z') = \int_0^{2\pi} d\theta' |\vec{r}-\vec{r}'| \sin \mu(\theta' - \theta). \quad (\text{F6})$$

One may show that  $I_2=0$  and  $I_1$  does not depend on the angle  $\theta$ . Hence (F5) is reduced to

$$I = I_1 \cos \mu\theta \quad (\text{F7})$$

thus delivering the desirable angular dependence in (F1).

Altogether, the Coulomb contribution (F1) to the response is recast to the form

$$U_k^{Coul}(\vec{r}) = U_k^{Coul}(\rho, z) \cos \mu\theta \quad (\text{F8})$$

with

$$\begin{aligned}
U_k^{Coul}(\rho, z) &= e^2 \int \rho' d\rho' dz' I_1(\rho, z; \rho', z') \cdot [\partial_{\rho'}^2 + \frac{1}{\rho'} \partial_{\rho'} - \frac{\mu^2}{(\rho')^2} + \partial_{z'}^2] \rho_{X,pk}(\rho', z') \\
&= e^2 \int \rho' d\rho' dz' \int_0^\pi d\theta_1 [(\rho + \rho')^2 + (z - z')^2 - 4\rho\rho' \cos^2 \theta_1]^{1/2} \cos 2\mu\theta_1 \\
&\quad \cdot [\partial_{\rho'}^2 + \frac{1}{\rho'} \partial_{\rho'} - \frac{\mu^2}{(\rho')^2} + \partial_{z'}^2] \rho_{X,pk}(\rho', z') .
\end{aligned} \tag{F9}$$

The matrix element (108) reads as

$$\langle i | \hat{X}_{pk}^p | j \rangle_{Coul} = e^2 \int \rho d\rho dz U_k^{Coul}(\rho, z) \rho_{ij}^s(\rho, z) \cdot \int_0^{2\pi} d\theta \cos \mu\theta e^{-i\mu\theta} . \tag{F10}$$

## APPENDIX G: MATRIX ELEMENTS

This Appendix represents the matrix elements of the operators  $\hat{P}_{sk}$ ,  $\hat{X}_{sk}^{s'}$  and  $\hat{Y}_{sk}^{s'}$ . The pairing factors are supposed to be included into the matrix elements.

### 1. Matrix elements of operator $\hat{P}_{sk}$

The matrix element of operator  $\hat{P}_{sk}$  reads

$$\begin{aligned}
\langle ij | P_{sk} | \tilde{0} \rangle &= 2i\epsilon_{ij} \langle ij | Q_{sk} | \tilde{0} \rangle - \langle ij | Y_{sk}^s | \tilde{0} \rangle \\
&= i\{ \overline{2\epsilon_{ij} \langle ij | Q_{sk} | \tilde{0} \rangle} - \overline{\langle ij | Y_{sk}^{s(1)} | \tilde{0} \rangle} (\delta_{\mu, K_i - K_j} + \delta_{-\mu, K_i - K_j}) \\
&\quad + \overline{\langle ij | Y_{sk}^{s(2)} | \tilde{0} \rangle} (\delta_{\mu, K_i - K_j} - \delta_{\mu, K_i - K_j}) \}
\end{aligned} \tag{G1}$$

for  $\mu = |K_i - K_j|$  and

$$\begin{aligned}
\langle i\bar{j} | P_{sk} | \tilde{0} \rangle &= 2i\epsilon_{ij} \langle i\bar{j} | Q_{sk} | \tilde{0} \rangle - \langle i\bar{j} | Y_{sk}^s | \tilde{0} \rangle \\
&= i\{ 2\epsilon_{ij} \overline{\langle i\bar{j} | Q_{sk} | \tilde{0} \rangle} - \overline{\langle i\bar{j} | Y_{sk}^{s(1)} | \tilde{0} \rangle} + \overline{\langle i\bar{j} | Y_{sk}^{s(2)} | \tilde{0} \rangle} \} \delta_{\mu, K_i + K_j}
\end{aligned} \tag{G2}$$

for  $\mu = K_i + K_j$ . Here,  $K_i$  and  $K_j$  are projections of the complete single-particle moment onto symmetry z-axis of the axial nucleus. Expressions for  $\langle ij | Y_{sk}^{s(1)} | \tilde{0} \rangle$  and  $\langle ij | Y_{sk}^{s(2)} | \tilde{0} \rangle$  are given in the section G 3. The combinations of Kronecker symbols follow from the angular integrals. Because of the specific combinations in (G1), all the terms of this matrix element have the same permutation properties for  $i \leftrightarrow j$ .

## 2. Matrix elements of operator $\hat{X}_{sk}^{s'}$

The matrix element of operator  $\hat{X}_{sk}^{s'}$  reads

$$\begin{aligned} \langle ij|\hat{X}_{sk}^{s'}|\tilde{0}\rangle &= \overline{\langle ij|\hat{X}_{sk}^{s'(1)}|\tilde{0}\rangle}(\delta_{\mu,K_i-K_j} + \delta_{\mu,K_j-K_i}) \\ &+ \overline{\langle ij|\hat{X}_{sk}^{s'(2)}|\tilde{0}\rangle}(\delta_{\mu,K_i-K_j} - \delta_{\mu,K_j-K_i}), \end{aligned} \quad (\text{G3})$$

for  $\mu = |K_i - K_j|$  and

$$\langle i\bar{j}|\hat{X}_{sk}^{s'}|\tilde{0}\rangle = \{\overline{\langle i\bar{j}|\hat{X}_{sk}^{s'(1)}|\tilde{0}\rangle} + \overline{\langle i\bar{j}|\hat{X}_{sk}^{s'(2)}|\tilde{0}\rangle}\}\delta_{\mu,K_i+K_j} \quad (\text{G4})$$

for  $\mu = K_i + K_j$ .

Here

$$\begin{aligned} \overline{\langle ij|\hat{X}_{sk}^{s'(1)}|\tilde{0}\rangle} &= \pi \int \rho d\rho dz [\bar{U}_{sk}^{s'}(\rho, z)\rho_{ij}^{s'}(\rho, z) + \bar{B}_{sk}^{s'}(\rho, z)\tau_{ij}^{s'}(\rho, z) \\ &+ \bar{W}_{sk;\rho}^{s'}(\rho, z)\mathfrak{S}_{ij;\rho}^{s'}(\rho, z) + \bar{W}_{sk;z}^{s'}(\rho, z)\mathfrak{S}_{ij;z}^{s'}(\rho, z)] \\ &+ \delta_{s,s'}\overline{\langle ij|\hat{X}_{sk}^{s'(pair)}|\tilde{0}\rangle} \end{aligned} \quad (\text{G5})$$

$$\overline{\langle ij|\hat{X}_{sk}^{s'(2)}|\tilde{0}\rangle} = \pi \int \rho d\rho dz \bar{W}_{sk;\theta}^{s'}(\rho, z)\mathfrak{S}_{ij;\theta}^{s'}(\rho, z), \quad (\text{G6})$$

$$\overline{\langle ij|\hat{X}_{sk}^{s(pair)}|\tilde{0}\rangle} = -\delta_{\mu,0}\delta_{i,j}G_s\chi_{X,ks}(u_i^2 - v_i^2). \quad (\text{G7})$$

The values entering (G7) are defined in Sec. II G.

Further

$$\begin{aligned} \bar{U}_{sk}^{s'}(\rho, z) &= \{b_0 - b'_0\delta_{s,s'} \\ &+ [-b_2 + b'_2\delta_{s,s'}][\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{\mu^2}{\rho^2} + \partial_z^2] + b_3\frac{(\alpha+1)(\alpha+2)}{3}\rho^\alpha(\rho, z) \\ &- b'_3[\frac{\alpha(\alpha-1)}{3}\rho^{\alpha-2}(\rho, z)\sum_{s''}\rho_{s''}^2(\rho, z) \\ &+ \frac{2}{3}\alpha\rho^{\alpha-1}(\rho, z)[\rho_s(\rho, z) + \rho_{s'}(\rho, z)] + \frac{2}{3}\rho^\alpha(\rho, z)\delta_{s,s'}] \\ &- \delta_{s',p}\delta_{s,p}\frac{e^2}{3}(\frac{3}{\pi})^{1/3}\rho_p^{-2/3}(\rho, z)\}\rho_{X,sk}(\rho, z) + \delta_{s',p}\delta_{s,p}U_k^{(\text{Coul})}(\rho, z) \\ &+ [b_1 - b'_1\delta_{s,s'}]\tau_{X,sk}(\rho, z) \\ &- [b_4 + b'_4\delta_{s,s'}][\frac{1}{\rho}\partial_\rho(\rho\mathfrak{S}_{X,sk}^\rho(\rho, z)) + \partial_z\mathfrak{S}_{X,sk}^z(\rho, z)] + \frac{\mu}{\rho}\mathfrak{S}_{X,sk}^\theta(\rho, z) \}, \end{aligned} \quad (\text{G8})$$

$$\bar{B}_{sk}^{s'}(\rho, z) = [b_1 - b'_1\delta_{s,s'}]\rho_{X,sk}(\rho, z), \quad (\text{G9})$$

$$\bar{W}_{sk;\rho}^{s'}(\rho, z) = [b_4 + b'_4 \delta_{s,s'}] \partial_\rho \rho_{X,sk}(\rho, z), \quad (\text{G10})$$

$$\bar{W}_{sk;z}^{s'}(\rho, z) = [b_4 + b'_4 \delta_{s,s'}] \partial_z \rho_{X,sk}(\rho, z), \quad (\text{G11})$$

$$\bar{W}_{sk;\theta}^{s'}(\rho, z) = -[b_4 + b'_4 \delta_{s,s'}] \frac{\mu}{\rho} \rho_{X,sk}(\rho, z). \quad (\text{G12})$$

The overline matrix elements are real. The combinations of Kronecker symbols follow from the angular integrals. Expressions for  $U_k^{(\text{Coul})}(\rho, z)$  in (G8) is done in (F9). The matrix elements (G4) and (G6) vanish at  $\mu = 0$ . Instead, the pairing matrix element (G7) exists only at  $\mu = 0$ .

### 3. Matrix elements of operator $\hat{Y}_{sk}^{s'}$

The matrix element of operator  $\hat{Y}_{sk}^{s'}$  reads

$$\begin{aligned} \langle ij | \hat{Y}_{sk}^{s'} | \tilde{0} \rangle &= \overline{i \langle ij | \hat{Y}_{sk}^{s'(1)} | \tilde{0} \rangle (\delta_{\mu, K_i - K_j} + \delta_{\mu, K_j - K_i})} \\ &\quad - \overline{\langle ij | \hat{Y}_{sk}^{s'(2)} | \tilde{0} \rangle (\delta_{\mu, K_i - K_j} - \delta_{\mu, K_j - K_i})}, \end{aligned} \quad (\text{G13})$$

for  $\mu = |K_i - K_j|$  and

$$\langle i\bar{j} | \hat{Y}_{sk}^{s'} | \tilde{0} \rangle = i \{ \overline{\langle i\bar{j} | \hat{Y}_{sk}^{s'(1)} | \tilde{0} \rangle} - \overline{\langle i | \hat{Y}_{sk}^{s'(2)} | j \rangle} \} \delta_{\mu, K_i + K_j} \quad (\text{G14})$$

for  $\mu = K_i + K_j$ . The overline matrix elements are real.

Further

$$\overline{\langle ij | \hat{Y}_{sk}^{s'(1)} | \tilde{0} \rangle} = \pi \int \rho d\rho dz \quad (\text{G15})$$

$$\cdot [\bar{A}_{sk;\rho}^{s'}(\rho, z) j_{ij;\rho}^{s'}(\rho, z) + \bar{A}_{sk;z}^{s'}(\rho, z) j_{ij;z}^{s'}(\rho, z) + \bar{S}_{sk;\theta}^{s'}(\rho, z) s_{ij;\theta}^{s'}(\rho, z)],$$

$$\overline{\langle i\bar{j} | \hat{Y}_{sk}^{s'(2)} | \tilde{0} \rangle} = \pi \int \rho d\rho dz \quad (\text{G16})$$

$$\cdot [\bar{A}_{sk;\theta}^{s'}(\rho, z) j_{ij;\theta}^{s'}(\rho, z) + \bar{S}_{sk;\rho}^{s'}(\rho, z) s_{ij;\rho}^{s'}(\rho, z) + \bar{S}_{sk;z}^{s'}(\rho, z) s_{ij;z}^{s'}(\rho, z)].$$

and

$$\bar{A}_{sk;\rho}^{s'}(\rho, z) = 2[-b_1 + b'_1 \delta_{s,s'}] j_{Y,sk;\rho}(\rho, z) \quad (\text{G17})$$

$$- [b_4 + b'_4 \delta_{s,s'}] \left[ \frac{\mu}{\rho} s_{Y,sk;z}(\rho, z) - \partial_z s_{Y,sk;\theta}(\rho, z) \right],$$

$$\bar{A}_{sk;z}^{s'}(\rho, z) = 2[-b_1 + b'_1 \delta_{s,s'}] j_{X,sk;z} \quad (\text{G18})$$



$$\begin{aligned}
& - [b_4 + b'_4 \delta_{s,s'}] \left[ \frac{1}{\rho} \partial_\rho (\rho s_{Y,sk;\theta}) - \frac{\mu}{\rho} s_{Y,sk;\rho} \right] \\
\bar{A}_{sk;\theta}^{s'}(\rho, z) &= 2[-b_1 + b'_1 \delta_{s,s'}] j_{Y,sk;\theta}
\end{aligned} \tag{G19}$$

$$\begin{aligned}
& - [b_4 + b'_4 \delta_{s,s'}] [\partial_z s_{Y,sk;\rho} - \partial_\rho s_{Y,sk;z}] , \\
\bar{S}_{sk;\rho}^{s'}(\rho, z) &= [b_4 + b'_4 \delta_{s,s'}] \left[ \frac{\mu}{\rho} j_{Y,sk;z}(\rho, z) + \partial_z j_{Y,sk;\theta}(\rho, z) \right] ,
\end{aligned} \tag{G20}$$

$$\bar{S}_{sk;z}^{s'}(\rho, z) = -[b_4 + b'_4 \delta_{s,s'}] \left[ \frac{1}{\rho} \partial_\rho (\rho j_{Y,sk;\theta}) + \frac{\mu}{\rho} j_{Y,sk;\rho} \right] , \tag{G21}$$

$$\bar{S}_{sk;\theta}^{s'}(\rho, z) = -[b_4 + b'_4 \delta_{s,s'}] [\partial_z j_{Y,sk;\rho} - \partial_\rho j_{Y,sk;z}] . \tag{G22}$$

For  $\mu = 0$ , one puts

$$\bar{A}_{sk;\theta}^{s'}(\rho, z) = \bar{S}_{sk;\rho}^{s'}(\rho, z) = \bar{S}_{sk;z}^{s'}(\rho, z) = 0 \tag{G23}$$

since these terms adjoin  $\sin \mu \theta$ . The remaining terms are reduced to

$$\bar{A}_{sk;\rho}^{s'}(\rho, z) = 2[-b_1 + b'_1 \delta_{s,s'}] j_{Y,sk;\rho}(\rho, z) - [b_4 + b'_4 \delta_{s,s'}] \partial_z s_{Y,sk;\theta}(\rho, z) , \tag{G24}$$

$$\bar{A}_{sk;z}^{s'}(\rho, z) = 2[-b_1 + b'_1 \delta_{s,s'}] j_{Y,sk;z} - [b_4 + b'_4 \delta_{s,s'}] \frac{1}{\rho} \partial_\rho (\rho s_{Y,sk;\theta}) , \tag{G25}$$

$$\bar{S}_{sk;\theta}^{s'}(\rho, z) = -[b_4 + b'_4 \delta_{s,s'}] [\partial_z j_{Y,sk;\rho} - \partial_\rho j_{Y,sk;z}] . \tag{G26}$$

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